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1

ON NORMAL FORMS OF DIFFERENTIAL EQUATIONS*

BY

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Klein† has treated the question of obtaining invariant forms for the differential equation

$$(1) \qquad y''+py'+qy = 0,$$

or the resolvent equation of the third order,

$$(2) \qquad [\eta] = 2q-p^2-\frac{dp}{dx},$$

where

$$[\eta] = \frac{\eta'''}{\eta'}-\frac{3}{2}\left(\frac{\eta''}{\eta'}\right)^2$$

is the Schwarzian derivative and the coefficients are single-valued functions on a given algebraic configuration; and he has given the solution for the hyperelliptic case, and the case of a canonical Riemann surface.

He raised the question of what the form would be in the case of a plane non-singular quartic, $p=3$, considered in the projective plane, when one imposes the further condition that the answer shall be given in terms of invariant expressions which bear symmetrically on the three ternary homogeneous variables x_1, x_2, x_3, and the question was answered by Gordan.‡

This last restriction, though doubtless interesting, is not prescribed by the nature of the problem, which admits an altogether satisfactory projective treatment for the case that one assumes the given algebraic configuration in the form of Noether's normal C_{p-1}. Let the projective homogeneous coördinates of a point of this curve be denoted by $(x_1, \cdots, x_p)$, and let the curve be projected on a pencil of hyperplanes,

$$(3) \qquad u_x - z v_x = 0,$$

* Presented to the Society, February 28, 1925.

† *Über lineare Differentialgleichungen der zweiten Ordnung*, Göttingen, 1894 (lithographed), pp. 90–105.

‡ Mathematische Annalen, vol. 46 (1895), p. 606. For further references to the literature cf. Fricke in the Encyklopädie der mathematischen Wissenschaften, vol. 2_2, pp. 437–8.

where

$$u_x = u_1 x_1 + \cdots + u_p x_p = 0, \qquad v_x = v_1 x_1 + \cdots + v_p x_p = 0$$

denote two non-specialized hyperplanes. Let F be the corresponding Riemann surface spread out over the z-plane, where

$$z = \frac{u_x}{v_x}.$$

Then F has $2p-2$ leaves, connected by $6p-6$ simple branch points.

The key to the solution of the problem, so far as the η-equation is concerned, consists in the identity*

$$[\eta]_z = \left(\frac{dt}{dz}\right)^2 [\eta]_t + [t]_z, \tag{4}$$

where

$$z = \varphi(t) \tag{5}$$

may be any analytic function whatever. Now choose, in particular, as the function $\varphi(t)$ the automorphic function with limiting circle, which maps F on a fundamental domain $\mathfrak{F}$ of the automorphic group. Then the form of (4) which is useful in what follows is

$$[\eta]_t = \varphi'(t)^2 \{[\eta]_z - [t]_z\}. \tag{6}$$

1. **The η-Differential Equation.** We wish to consider such differential equations

$$[\eta] = \text{single valued function on } C_{p-1} \tag{A}$$

as have only *regular* singular points on C_{p-1}. These points, n in number, are given, and the difference of the exponents in each is given. In particular, there are ∞^{3p-3} equations (A) having no singular points. (Here, $p > 2$, since we will assume the C_{p-1} to be simple, and thus exclude the hyperelliptic case.)

The Manifold S_p. Let S_p denote the real four-dimensional manifold of the points $(x_1, \cdots, x_p)$ corresponding to Noether's C_{p-1}. It will be convenient to uniformize S_p as follows.† Let

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* Klein, loc. cit., p. 59.

† Cf. *The Madison Colloquium*, p. 224.

(7) $$w_k = \int \Phi_k(t)\,dt \qquad (k = 1, \cdots, p)$$

denote the normal integrals of the first kind, and let C_{p-1} be assumed in the form

$$C_{p-1}: \qquad x_1 : x_2 : \cdots : x_p = \Phi_1(t) : \Phi_2(t) : \cdots : \Phi_p(t).$$

Then we may set

(8) $$x_k = \varrho\,\Phi_k(t) \qquad (k = 1, \cdots, p).$$

If t be restricted to the fundamental domain $\mathfrak{F}$ and ϱ be allowed to take on any value but 0, then, not only will each pair of values (ϱ, t) lead to one point of S_p, but conversely each point of S_p will lead to just one such pair of values (ϱ, t). Furthermore, to an arbitrary point (x^0) of S_p will correspond at least one pair of integers (α, β)—which may be different for a second point (x^1) of S_p—such that the equations

$$x_\alpha = \varrho\,\Phi_\alpha(t), \qquad x_\beta = \varrho\,\Phi_\beta(t),$$

when solved for ϱ and t, yield functions

$$\varrho = \varrho(x_\alpha, x_\beta), \qquad t = t(x_\alpha, x_\beta)$$

both analytic in (x_α, x_β) in the neighborhood of the point (x_α^0, x_β^0).

The Manifold Σ_p. We proceed to introduce a new four-dimensional manifold Σ_p corresponding to the Riemann surface F spread out over the z-plane, where

(9) $$z = \frac{u_x}{v_x} = \frac{u_\Phi}{v_\Phi}.$$

Let

(10) $$z_1 = u_x, \qquad z_2 = v_x.$$

Then Σ_p is the Riemann manifold whose points are (z_1, z_2), and it is uniformized by the equations

(11) $$\begin{aligned} z_1 &= \varrho u_\Phi = \varrho[u_1\Phi_1(t) + \cdots + u_p\Phi_p(t)], \\ z_2 &= \varrho v_\Phi = \varrho[v_1\Phi_1(t) + \cdots + v_p\Phi_p(t)]. \end{aligned}$$

The branch function, s, on F is given by the formula

$$s = \frac{dz}{dW}, \qquad W = \int v_\Phi \, dt.$$

Thus

$$s = \frac{v_\Phi u_{\Phi'} - u_\Phi v_{\Phi'}}{v_\Phi^3}. \tag{12}$$

The branch form, σ, is now defined by the equation

$$\sigma = \varrho^3 (v_\Phi u_{\Phi'} - u_\Phi v_{\Phi'}). \tag{13}$$

Let the last factor be denoted by D, or $D(t)$. Then

$$\sigma = \varrho^3 D. \tag{14}$$

We now have the material in hand for writing the η-differential equation in the desired form. The function $\varphi(t)$ which we chose in equation (6) is no other than the function defined by (9):

$$\varphi(t) = \frac{u_\Phi}{v_\Phi}.$$

Thus

$$\varphi'(t) = \frac{D}{v_\Phi^2} = \frac{\sigma}{\varrho z_2^2}.$$

Moreover, Klein has pointed out that the form

$$\frac{1}{z_2^4} [\eta]_z$$

is invariant under a linear transformation of the binary homogeneous variables z_1, z_2. Hence we write (6) in the form

$$\sigma^2 \left\{ \frac{1}{z_2^4} [\eta]_z - \frac{1}{z_2^4} [t]_z \right\} = \varrho^2 [\eta]_t. \tag{15}$$

From this identity follow the two normal forms of the η-equation, which we set out to obtain, namely

$$(A_1) \qquad \sigma^2 \left\{ \frac{1}{z_2^4} [\eta]_z - \frac{1}{z_2^4} [t]_z \right\} = \Gamma(x_1, x_2, \cdots, x_p),$$

where Γ denotes a homogeneous rational function of the second dimension, whose singularities form precisely the singular points of the η-equation

$$(A_2) \qquad [\eta]_t = Q(t),$$

where

$$Q(t) = \frac{\Gamma(\varrho\Phi_1(t), \cdots, \varrho\Phi_p(t))}{\varrho^2}.$$

Thus $Q(t)$ is single-valued and meromorphic. It is not an absolute invariant of the automorphic group, but takes on a factor for each transformation

$$(16) \qquad t_\alpha = L_\alpha(t)$$

of this group. Since each x_k is invariant under (16), we have

$$\varrho_\alpha \Phi_k(t_\alpha) = \varrho \Phi_k(t).$$

Moreover,

$$du_k = \Phi_k(t_\alpha)dt_\alpha = \Phi_k(t)dt.$$

Hence

$$(17) \qquad \varrho_\alpha = L'_\alpha(t)\varrho, \qquad \Phi_k(t_\alpha) = L'^{-1}_\alpha(t)\Phi_k(t).$$

Thus it appears that

$$(18) \qquad Q(t_\alpha) = L'_\alpha(t)^{-2} Q(t).$$

2. **Discussion of** (A_1) **and** (A_2). Equation (A_1), which may be called the *algebraic* form of the η-differential equation, is based on the algebraic manifold assumed in the form of Noether's C_{p-1}. This real two-dimensional manifold is replaced by the four-dimensional manifold S_p of the homogeneous variables $(x_1, \cdots, x_p)$. The latter manifold, like the former, has no singular points whatever. Let (a) be a point of S_p in which the η-differential equation is to have a regular singular point, and let the difference of the exponents there be denoted by α. Now let u_x and v_x be so chosen that

(i) the hyperplane $v_x = 0$ does not pass through (a);

(ii) the branch form σ does not vanish in (a).

Then (a) will go over into a finite point $z = a$ of F, which is not a branch point. The function $[t]_z$ will be analytic at this point. On the other hand,

$$[\eta]_z = \frac{(1-\alpha^2)/2}{(z-a)^2} + \frac{C}{z-a} + \mathfrak{A}(z),$$

where $\mathfrak{A}(z)$ is a generic notation for a function analytic at $z = a$.

It follows, then, from (A_1) that $\Gamma(x_1, \cdots, x_p)$ must have a pole of the second order in (a), and that the coefficient of the term of the second order in the principal part of this pole is determined.

Since the number of singular points of the differential equation is finite, u_x and v_x can be so chosen that conditions (i) and (ii) will be satisfied for every singular point of the differential equation. Similarly, u_x and v_x can be so chosen that an arbitrary point (b) of S_p will go over into a finite point of F, not a branch point; and since both $[\eta]_z$ and $[t]_z$ will be analytic there if (b) is not a singular point of the differential equation, it follows that $\Gamma(x_1, \cdots, x_p)$ is analytic at (b).

But u_x and v_x cannot be so chosen once for all that all of the above conditions will be fulfilled for every point of S_p. It is like the case of the normal differential on a non-singular curve of the projective (x_1, x_2, x_3)-plane:

$$d\omega = \frac{\begin{vmatrix} c_1 & x_1 & dx_1 \\ c_2 & x_2 & dx_2 \\ c_3 & x_3 & dx_3 \end{vmatrix}}{c_1 f_1 + c_2 f_2 + c_3 f_3}.$$

If a point (a) of the curve be chosen in advance, then (c) can be so taken that the denominator does not vanish in (a). But (c) cannot be so taken once for all that this condition is fulfilled for every point of the curve.

Equation (A_2). This form, which may be called the *automorphic* form of the η-differential equation, meets completely the difficulty just discussed, for it holds without let or hindrance for *every* point in $\mathfrak{F}$ and its analytic continuations. The function $Q(t)$ is single-valued and analytic in every point t except the singular points of the differential equation. It is the form which serves as the definition of η when this function is studied on the basis of $\mathfrak{F}$ as the defining element of the given algebraic equation.

3. **The Linear Differential Equation.** It would seem to be a simple matter to pass from equation (A_2) to the linear differential equation corresponding to (1), since one need only set

$$(19) \qquad y_1 = \frac{\eta}{\sqrt{\frac{d\eta}{dt}}}, \qquad y_2 = \frac{1}{\sqrt{\frac{d\eta}{dt}}}.$$

and these functions are linearly independent solutions of the equation

$$\frac{d^2 y}{dt^2} + \tfrac{1}{2} Q(t)\, y = 0. \tag{20}$$

And, indeed, for the automorphic treatment this is the whole story.

When, however, these functions y_1 and y_2 are transplanted to the algebraic form C_{p-1} or S_p of the algebraic configuration, they do not satisfy a differential equation of the form (1),—namely, one whose coefficients are single-valued on C_{p-1} or S_p. In fact,

$$\frac{d\eta}{dt} \quad \text{and} \quad \frac{d\eta}{dt_\alpha} = \frac{1}{L'_\alpha(t)} \frac{d\eta}{dt}$$

are two different functions, although t and $t_\alpha = L_\alpha(t)$ correspond to the same point of C_{p-1}.*

On the other hand, if the η-equation be assumed in the form arising from (A_1),

$$[\eta]_z - [t]_z = R(z, s), \tag{21}$$

and if now we set

$$Y_1 = \frac{\eta}{\sqrt{\dfrac{d\eta}{dz}}}, \qquad Y_2 = \frac{1}{\sqrt{\dfrac{d\eta}{dz}}}, \tag{22}$$

the points of the given configuration for which $z = \infty$, and also the branch points, assume an exceptional rôle.

How shall these two classes of difficulties be avoided? Klein answers the question by the use of homogeneous variables and transcendental forms. He sets

$$H_1 = \frac{\eta}{\sqrt{\dfrac{d\eta}{d\omega}}}, \qquad H_2 = \frac{1}{\sqrt{\dfrac{d\eta}{d\omega}}}, \tag{23}$$

where $d\omega$ is the *normal differential*, a so-called *differential form*, which, for the C_{p-1}—or rather for the S_p—is defined as follows:

$$d\omega = \frac{|z\,dz|}{\sigma}, \qquad |z\,dz| = z_2\, dz_1 - z_1\, dz_2. \tag{24}$$

* The same conclusion may be reached by observing that the coefficients of (20) are not single-valued on F, C_{p-1}, or S_p, whereas this differential equation is uniquely determined from (A_2).

The Normal Differential. No matter what the independent variable or variables may be, equations (11) give in all cases

$$\begin{vmatrix} dz_1 & dz_2 \\ z_1 & z_2 \end{vmatrix} = \begin{vmatrix} \varrho u_{\Phi'}\, dt + u_\Phi\, d\varrho & \varrho v_{\Phi'}\, dt + v_\Phi\, d\varrho \\ \varrho u_\Phi & \varrho v_\Phi \end{vmatrix} = \varrho^2 D\, dt.$$

Hence

$$\frac{|z\, dz|}{\sigma} = \frac{\varrho^2 D\, dt}{\varrho^3 D}$$

and

$$d\omega = \frac{dt}{\varrho}. \tag{25}$$

Thus*

$$\frac{d\eta}{d\omega} = \varrho \frac{d\eta}{dt},$$

and

$$\Pi_1 = \varrho^{-1/2} \frac{\eta}{\sqrt{\dfrac{d\eta}{dt}}}, \qquad \Pi_2 = \varrho^{-1/2} \frac{1}{\sqrt{\dfrac{d\eta}{dt}}}. \tag{26}$$

The Differential Equation for Π. The expressions Π_1, Π_2 are *transcendental forms* (i. e. homogeneous functions) of dimension $-\frac{1}{2}$, in z_1, z_2. They satisfy a differential equation of the following form:†

$$(\Pi, \sigma^2)_2 + 15 \left\{ \frac{\tau}{\sigma} + \Gamma \right\} \Pi = 0, \tag{B}$$

where the first term denotes the second transvectant of Π and σ^2, and τ is an integral algebraic form of the fifth dimension, belonging to S_p. The proof follows.

4. **Deduction of (B).** The second transvectant $(\Pi, \varphi)_2$ of two binary forms, Π and φ, can be expressed as follows:

$$(\Pi, \varphi)_2 = \varphi_{22} \Pi_{11} - 2 \varphi_{12} \Pi_{12} + \varphi_{11} \Pi_{22}. \tag{27}$$

* The expression $d^2\eta/d\omega^2$ could be defined for a thread, or for the case that ρ and t are both analytic functions of a third complex variable. It appears to be useless,—I know, at least, of no place in which Klein considers it.

† Klein, loc. cit., p. 98.

The partial derivatives of a form ψ of dimension k, with respect to z_2, are given by the formulas (Klein, loc. cit., pp. 23, 24)

$$\psi_2 = \frac{k\psi - z_1\psi_1}{z_2}, \qquad \psi_{12} = \frac{(k-1)\psi_1 - z_1\psi_{11}}{z_2^2}. \tag{28}$$
$$\psi_{22} = \frac{k(k-1)\psi - 2(k-1)z_1\psi_1 + z_1^2\psi_{11}}{z_2^2}.$$

Since, in (27), H and $\varphi = \sigma^2$ are of dimension $-\frac{1}{2}$ and 6 respectively, we have

$$z_2^2(H, \sigma^2)_2 = 30\varphi H_{11} + 15\varphi_1 H_1 + \tfrac{3}{4}\varphi_{11} H. \tag{29}$$

We proceed to compute the right hand side of (29) in terms of ϱ and t, where H denotes either one of the functions (23), and y, the corresponding function (19). Thus

$$H = \varrho^{-1/2} y \qquad \text{and} \qquad \varphi = \sigma^2 = \varrho^6 D^2. \tag{30}$$

For brevity, we write

$$u_\Phi = u, \qquad u_{\Phi'} = u', \qquad \text{etc.}$$

Thus

$$z_1 = \varrho u, \qquad z_2 = \varrho v.$$

Hence

$$\frac{\partial \varrho}{\partial z_1} = -\frac{v'}{D}, \qquad \frac{\partial t}{\partial z_1} = \frac{v}{\varrho D}. \tag{31}$$

The computation yields the following values:

$$H_1 = \varrho^{-3/2}\left\{\frac{v}{D}y' + \frac{1}{2}\frac{v'}{D}y\right\},$$
$$H_{11} = \varrho^{-5/2}\left\{\frac{v^2}{D^2}y'' + \frac{v(3v'D - vD')}{D^3}y' + \frac{\frac{3}{4}v'^2 D + \frac{1}{2}vv''D - \frac{1}{2}vv'D'}{D^3}y\right\}, \tag{32}$$
$$\varphi_1 = \varrho^5\{2vD' - 6v'D\},$$
$$\varphi_{11} = \varrho^4\left\{2v^2\frac{D''}{D} - 14vv'\frac{D'}{D} + 30v'^2 - 6vv''\right\}.$$

On substituting these in (29) and reducing, we have

$$(33)\qquad (\Pi, \sigma^2)_2 = 15\varrho^{8/2}\{2y'' - \varrho^2 T y\},$$

$$T = -\frac{1}{10}\left\{\frac{D''}{D} + 7\frac{v''}{v} - 7\frac{v'D'}{vD}\right\}.$$

From (20) and (30) it follows, since

$$Q(t) = \frac{\Gamma(x_1, \cdots, x_p)}{\varrho^2},$$

that

$$(34)\qquad (\Pi, \sigma^2)_2 + 15\{\varrho^2 T + \Gamma(x)\}\Pi = 0,$$

and it remains to discuss the form T.

5. **The Forms** T, τ. It is to be observed that (33) is an identity in the sense that y may be any analytic function of t whatever, Π being then determined by (30); and that T is independent of y. It is possible, therefore, so to choose y that y and Π will be analytic in each point which corresponds to a root of v, and that y will not vanish there. Hence T must remain finite there, and this fact suggests that the terms of the brace,

$$7\frac{v''}{v} - 7\frac{v'D'}{vD} = 7\frac{v''D - v'D'}{vD},$$

admit an algebraic reduction. In fact, since

$$D = vu' - v'u$$

and

$$D' = vu'' - v''u$$

it appears that

$$v''D - v'D' = v(v''u' - v'u''),$$

and thus we have

$$(35)\qquad T = -\frac{1}{10}\,\frac{D'' + 7(v''u' - v'u'')}{D}.$$

The two-rowed determinants which enter suggest the following notation:

$$\begin{vmatrix} u^{(j)} & v^{(j)} \\ u^{(k)} & v^{(k)} \end{vmatrix} = |u^j v^k|. \tag{36}$$

Thus we have finally

$$T = -\frac{1}{10}\frac{|u^3 v^0| - 6|u^2 v^1|}{|u^1 v^0|}. \tag{37}$$

Invariant Property. From (34) we readily conjecture that $\varrho^2 T$, which is a form of dimension 2, is invariant under the automorphic group; i.e., if

$$t_\alpha = L_\alpha(t)$$

be a transformation of that group, then

$$T(t_\alpha) = L_\alpha'^{-2}(t)\,T(t). \tag{38}$$

The correctness of this surmise is readily proved by direct computation. Let

$$L_\alpha(t) = \frac{at+b}{ct+d}.$$

Then

$$L'_\alpha(t) = \frac{\Delta}{(ct+d)^2}, \qquad \Delta = ad - bc.$$

For convenience, let $\Delta = 1$. Furthermore, from (17),

$$v_\alpha = v(t_\alpha) = L_\alpha'^{-1}(t)v(t) = (ct+d)^2 v.$$

Hence

$$v'(t_\alpha)\,dt_\alpha = \frac{d}{dt}\{(ct+d)^2 v\}\,dt,$$

or

$$v'_\alpha = v'(t_\alpha) = (ct+d)^4 v' + 2c(ct+d)^3 v,$$

with like formulas for v''_α and v'''_α.

On substituting in (37), (38) results.

The Form τ. Let τ be defined by the equation

$$\begin{aligned} \tau &= -\frac{1}{10}\varrho^5\{D'' + 7(v''u' - v'u'')\} \\ &= -\frac{1}{10}\varrho^5\{|u^3v^0| - 6|u^2v^1|\}. \end{aligned} \tag{39}$$

Then τ is seen to be an integral algebraic form of dimension 5 on either Σ_p or S_p, and

$$\varrho^2 T = \frac{\tau}{\sigma}. \tag{40}$$

Thus (B) is established.

The Function $[t]_z$ *in Terms of* t. We append the following formula:

$$\begin{aligned}[t]_z &= \frac{v^4}{D^3}\left\{2(v''u' - v'u'') - D'' + \frac{3}{2}\frac{D'^2}{D}\right\} \\ &= \frac{v^4}{D^3}\left\{\frac{3}{2}\frac{D'^2}{D} - [\,|u^3 v^0| + 3|u^2 v^1|\,]\right\}.\end{aligned} \tag{41}$$

6. **Computation of** τ **in** z_1, z_2. If we set Y equal to either of the functions (22), then

$$Y = \sqrt{\frac{dz}{dt}}\, y, \qquad \frac{dz}{dt} = \frac{D}{v^2},$$

and

$$H = \frac{z_2}{\sqrt{\sigma}}\, Y. \tag{42}$$

By computation similar to that of § 4 an identity analogous to (33) is obtained, namely

$$(H, \sigma^2)_2 = 30\frac{\sigma^{3/2}}{z_2^3}\, Y'' + \frac{9\sigma_1^2 - \frac{27}{2}\sigma\sigma_{11}}{z_2\,\sigma^{1/2}}\, Y. \tag{43}$$

On the other hand from (A_1), written in the form

$$[\eta]_z = [t]_z + \frac{z_2^4}{\sigma^2}\, \Gamma(x),$$

follows the equation (1) for Y, namely,

$$\frac{d^2 Y}{dz^2} + \left\{\frac{1}{2}[t]_z + \frac{1}{2}\frac{z_2^4}{\sigma^2}\,\Gamma(x)\right\} Y = 0. \tag{44}$$

From (42), (43), and (44) we now infer

$$(45)\qquad (H, \sigma^2)_2 + 15\left\{\frac{\sigma^2}{z_2^4}[t]_z - \frac{6\sigma_1^2 - 9\sigma\sigma_{11}}{10 z_2^2} + I(x)\right\} H = 0.$$

On comparing (45) with (B) we see that τ has the value

$$(46)\qquad \tau = \frac{\sigma^3}{z_2^4}[t]_z - \frac{\sigma(6\sigma_1^2 - 9\sigma\sigma_{11})}{10 z_2^2}.$$

Remark. It appears that τ, like σ, is an integral algebraic form belonging to the manifold Σ_p and uniquely determined by it. In terms of σ and τ the differential equation for t as a function on Σ_p assumes the form

$$(47)\qquad \frac{1}{z_2^4}[t]_z = \frac{\tau}{\sigma^3} + \frac{6\sigma_1^2 - 9\sigma\sigma_{11}}{10 z_2^2 \sigma^2}.$$

7. **The Parameters t and ϱ as Functions on Σ_p.** In the theory of the differential equations (1) and (2) it is a leading question to find conditions of scientific importance which determine uniquely the differential equation. One answer to this question was given by Klein through the method of conformal mapping* and consists in the case before us in the function t,—the inverse of the function (6), $z = \varphi(t)$,—and the differential equation which it satisfies,

$$(48)\qquad [t]_z = \varpi(z, s).$$

For t is uniquely determined save as to a linear transformation. The Schwarzian derivative is invariant of a linear transformation of t. Hence $\varpi(z, s)$ is uniquely determined on F, and thus becomes a function belonging to the surface. This function can be expressed in terms of σ and τ by means of (47):

$$(49)\qquad \varpi(z, s) = \frac{z_2^4}{\sigma^2}\left\{\frac{\tau}{\sigma} + \frac{6\sigma_1^2 - 9\sigma\sigma_{11}}{10 z_2^2}\right\}.$$

* When the linear differential equation (2) is considered in the real domain, the theorems of oscillation can be used for a similar purpose.

Thus the parameter t admits the interpretation of being a solution of the differential equation on F given by Formula (48).

The other parameter, ϱ, can be interpreted by means of the linear differential equation of the second order for Π, Formula (B). If in (26) we set $\eta = t$, then one of the functions Π reduces to $\varrho^{-1/2}$. On the other hand, if we set $\eta = t$ in (A_2), $Q(t)$ vanishes identically; hence also $\Gamma(x)$. Thus the equation (B) which corresponds to $\eta = t$ reduces to

$$(\Pi, \sigma^2)_2 + 15 \frac{\tau}{\sigma} \Pi = 0, \tag{50}$$

and one solution of this equation is

$$\Pi = \varrho^{-1/2}.$$

Harvard University,
Cambridge, Mass.
May, 1924.

CONGRUENCES WITH CONSTANT ABSOLUTE INVARIANTS*

BY

H. L. OLSON

I. Introduction

It is well known that any congruence can be regarded as the aggregate of lines tangent to two surfaces, or, as some authors prefer to say, the double tangent lines to a surface of two sheets called the focal surface. The present discussion will deal only with a portion of a congruence in which each line touches the surface in two distinct points. We shall make the further assumption that neither sheet of the focal surface is developable or degenerates into a curve.

Wilczynski has shown that the homogeneous coördinates, y_1, y_2, y_3, y_4 and z_1, z_2, z_3, z_4, respectively, of the points in which a line of the congruence touches the two sheets, S_y and S_z, of the focal surface, may be taken as four linearly independent solutions of a system of partial differential equations of the form

$$(D)\qquad \begin{aligned} y_v &= mz, \qquad z_u = ny, \\ y_{uu} &= ay + bz + cy_u + dz_v, \\ z_{vv} &= a'y + b'z + c'y_u + d'z_v. \end{aligned}$$

Since, when four linearly independent solutions, $y_i, z_i (i = 1, 2, 3, 4)$, are known, any four linearly independent linear combinations of these (with constant coefficients) can be taken as a fundamental system, the system of differential equations (D) can be regarded as representing the totality of congruences projective to a given one.

In order to obtain a system of equations of the form (D) it is necessary, besides taking the loci of y and z to be the two sheets of the focal surface, to choose the independent variables, u and v, so that if u be taken constant the variable line yz will in every case generate a developable having its cuspidal edge on S_y and if v be taken constant the line yz will in every case generate a developable having its cuspidal edge on S_z.

In order that this system of differential equations may have four linearly independent solutions (y, z) certain restrictions must be placed upon the coefficients; in the first place we must have $c_v = d'_u$; in other words, there must exist a function f such that

$$(1)\qquad c = f_u, \qquad d' = f_v.$$

* Presented to the Society, February 28, 1925.

The following further conditions must be satisfied:

$$
(2) \qquad
\begin{aligned}
b &= -d_v - d f_v, \qquad a' = -c'_u - c' f_u, \\
mn - c' d &= f_{uv} = W, \\
m_{uu} + d_{vv} + d f_{vv} + d_v f_v - f_u m_u &= m a + d b', \\
n_{vv} + c'_{uu} + c' f_{uu} + c'_u f_u - f_v n_v &= c' a + n b', \\
2 m_u n + m n_u &= a_v + f_u m n + a' d, \\
m_v n + 2 m n_v &= b'_u + f_v m n + b c'.
\end{aligned}
$$

It is geometrically evident that the most general transformation under which the above-mentioned geometrical properties are preserved is

$$
(3) \qquad y = \lambda(u, v)\,\bar{y}, \qquad z = \mu(u, v)\,\bar{z},
$$

$$
(4) \qquad \bar{u} = \alpha(u), \qquad \bar{v} = \beta(v).
$$

The transformed equations will have the particular form (D) if and only if λ is a function of u only and μ is a function of v only,

$$
(5) \qquad y = \lambda(u)\,\bar{y}, \qquad z = \mu(v)\,\bar{z}.
$$

Under transformations of the types (4) and (5), certain functions of the coefficients and their derivatives are unchanged except perhaps for multiplication by functions of α, β, λ, and μ and their derivatives; such functions are called relative invariants. An invariant which is absolutely unchanged by transformations of the types (4) and (5) is called an absolute invariant. A fundamental set of relative invariants, i. e., a set having the property that every relative invariant is expressible in terms of these and their derivatives, consists of m, n, c', d, and

$$
(6) \qquad
\begin{aligned}
\overset{(y)}{\mathfrak{B}} &= \frac{f_u}{4} - \frac{1}{8}\frac{\partial}{\partial u}\log(d m^3), \\
\overset{(z)}{\mathfrak{B}} &= \frac{f_u}{4} + \frac{1}{8}\frac{\partial}{\partial u}\log(c'^3 n), \\
\overset{(y)}{\mathfrak{C}}{}'' &= \frac{f_v}{4} + \frac{1}{8}\frac{\partial}{\partial v}\log(d^3 m), \\
\overset{(z)}{\mathfrak{C}}{}'' &= \frac{f_v}{4} - \frac{1}{8}\frac{\partial}{\partial v}\log(c' n^3).
\end{aligned}
$$

In defining these invariants (6) we assume, evidently, that m, n, c', and d are different from 0 for the values of u and v considered. It is the purpose

of this thesis to study the properties of those congruences of which the absolute invariants are constants. Under the transformations (4) and (5) the relative invariants mentioned above become*

$$
(7)\qquad
\begin{aligned}
&\bar{m} = \frac{\mu}{\lambda\beta_v}\, m, \quad \bar{n} = \frac{\lambda}{\mu\alpha_u}\, n, \quad \bar{c}' = \frac{\lambda\alpha_u}{\mu\beta_v^2}\, c', \quad \bar{d} = \frac{\mu\beta_v}{\lambda\alpha_u^2}\, d,\\
&\overline{\overset{(y)}{\mathfrak{B}}} = \frac{1}{\alpha_u}\overset{(y)}{\mathfrak{B}}, \quad \overline{\overset{(z)}{\mathfrak{B}}} = \frac{1}{\alpha_u}\overset{(z)}{\mathfrak{B}}, \quad \overline{\overset{(y)}{\mathfrak{C}}}{}'' = \frac{1}{\beta_v}\overset{(y)}{\mathfrak{C}}{}'', \quad \overline{\overset{(z)}{\mathfrak{C}}}{}'' = \frac{1}{\beta_v}\overset{(z)}{\mathfrak{C}}{}''.
\end{aligned}
$$

We shall need some further material from Wilczynski's Brussells paper: the differential equations of the two sheets of the focal surface are, respectively,

$$
(8)\qquad
\begin{aligned}
m\, y_{uu} - d\, y_{vv} &= a\, m\, y + c\, m\, y_u + \left(b - \frac{d\, m_v}{m}\right) y_v,\\
y_{uv} &= m\, n\, y + \frac{m_u}{m}\, y_v,
\end{aligned}
$$

and

$$
(9)\qquad
\begin{aligned}
-c'\, z_{uu} + n\, z_{vv} &= b'\, n\, z + \left(a' - \frac{c'\, n_u}{n}\right) z_u + d'\, n\, z_v,\\
z_{uv} &= m\, n\, z + \frac{n_v}{n}\, z_u.\dagger
\end{aligned}
$$

The differential equations of the 1st and (—1)st Laplace transforms are

$$
(10)\qquad
\begin{aligned}
y_v^{(1)} &= m^{(1)} z^{(1)}, \qquad z_u^{(1)} = n^{(1)} y^{(1)},\\
y_{uu}^{(1)} &= a^{(1)} y^{(1)} + b^{(1)} z^{(1)} + c^{(1)} y_u^{(1)} + d^{(1)} z_v^{(1)},\\
z_{vv}^{(1)} &= a'^{(1)} y^{(1)} + b'^{(1)} z^{(1)} + c'^{(1)} y_u^{(1)} + d'^{(1)} z_v^{(1)},
\end{aligned}
$$

and a similar system with the superfix (—1) instead of (1), where

$$
(11)\qquad
\begin{aligned}
m^{(1)} &= m\left(m\, n - \frac{\partial^2}{\partial u\, \partial v} \log m\right),\\
n^{(1)} &= \frac{1}{m}.
\end{aligned}
$$

* Wilczynski, *Sur la théorie générale des congruences*, pp. 9-20 (cited hereafter as Brussells paper).

† Brussells paper, Section 7.

$$a^{(1)} = a + f_{uu} - 2\frac{\partial^2}{\partial u^2}\log m + \left(\frac{d_u}{d} - \frac{m_u}{m}\right)\left(\frac{m_u}{m} - f_u\right).$$

$$\begin{aligned} b^{(1)} = {} & m\left(a_u + bn + af_u + dn_v - \frac{\partial^3}{\partial u^3}\log m\right) \\ & + \frac{m_v}{m}\left(b_u + bf_u + dmn - \frac{bm_u}{m}\right) \\ & + m_u\left(f_{uu} + f_u^2 - f_u\frac{m_u}{m} - 2\frac{\partial^2}{\partial u^2}\log m\right) \\ & + \left(f_u + \frac{d_u}{d} - \frac{m_u}{m}\right)\left(m_{uu} - f_u m_u - \frac{bm_v}{m} - am\right). \end{aligned} \tag{11}$$

$$c^{(1)} = f_u + \frac{d_u}{d} - \frac{m_u}{m}.$$

$$d^{(1)} = d\left(c'd - \frac{\partial^2}{\partial u\,\partial v}\log d\right).$$

$$a'^{(1)} = \frac{1}{d}\left(\frac{m_u}{m} - f_u\right).$$

$$b'^{(1)} = -\frac{\partial^2}{\partial v^2}\log m + \frac{1}{d}\left(m_{uu} - am - f_u m_u - \frac{bm_v}{m}\right).$$

$$c'^{(1)} = \frac{1}{d}.$$

$$d'^{(1)} = f_v + \frac{d_v}{d} - \frac{m_v}{m}.$$

and

$$m^{(-1)} = \frac{1}{n},$$

$$n^{(-1)} = n\left(mn - \frac{\partial^2}{\partial u\,\partial v}\log n\right).$$

$$\begin{aligned} a^{(-1)} &= -\frac{\partial^2}{\partial u^2}\log n + \frac{1}{c'}\left(n_{vv} - b'n - f_v n_v - \frac{a'n_u}{n}\right). \\ b^{(-1)} &= \frac{1}{c'}\left(\frac{n_v}{n} - f_v\right). \\ c^{(-1)} &= \frac{c'_u}{c'} - \frac{n_u}{n} + f_u, \\ d^{(-1)} &= \frac{1}{c'}, \end{aligned} \tag{12}$$

$$
\begin{aligned}
a'^{(-1)} = {} & n\left(b'_v + a'm + b'f_v + c'm_u - \frac{\partial^3}{\partial v^3}\log n\right) \\
& + \frac{n_u}{n}\left(a'_v + a'f_v + c'mn - \frac{a'n_v}{n}\right) \\
& + n_v\left(f_{vv} + f_v^2 - f_v\frac{n_v}{n} - 2\frac{\partial^2}{\partial v^2}\log n\right) \\
& + \left(f_v + \frac{c'_v}{c'} - \frac{n_v}{n}\right)\left(n_{vv} - f_v n_v - \frac{a'n_u}{n} - b'n\right). \\
b'^{(-1)} = {} & b' + f_{vv} - 2\frac{\partial^2}{\partial v^2}\log n + \left(\frac{c'_v}{c'} - \frac{n_v}{n}\right)\left(\frac{n_v}{n} - f_v\right). \\
c'^{(-1)} = {} & c'\left(c'd - \frac{\partial^2}{\partial u\,\partial v}\log c'\right). \\
d'^{(-1)} = {} & f_v + \frac{c'_v}{c'} - \frac{n_v}{n}.^{*}
\end{aligned}
\tag{12}
$$

II. Case 1, $\overset{(y)}{\mathfrak{B}} \not\equiv \overset{(z)}{\mathfrak{B}}$, $\overset{(y)}{\mathfrak{C}}{}'' \not\equiv \overset{(z)}{\mathfrak{C}}{}''$

1. Under these conditions it is convenient to use the following absolute invariants, which, as stated above, are assumed to be constants:

$$
\begin{aligned}
i_1 &= \frac{c'd}{mn}, \\
i_2 &= \frac{\overset{(y)}{\mathfrak{B}}}{\overset{(z)}{\mathfrak{B}} - \overset{(y)}{\mathfrak{B}}} = \frac{2f_u - \dfrac{\partial}{\partial u}\log(dm^3)}{\dfrac{\partial}{\partial u}\log(c'^3 dm^3 n)}, \\
i_3 &= \frac{\overset{(z)}{\mathfrak{C}}{}''}{\overset{(y)}{\mathfrak{C}}{}'' - \overset{(z)}{\mathfrak{C}}{}''} = \frac{2f_v - \dfrac{\partial}{\partial v}\log(c'n^3)}{\dfrac{\partial}{\partial v}\log(c'd^3mn^3)}, \\
i_4 &= \frac{\overset{(z)}{\mathfrak{B}} - \overset{(y)}{\mathfrak{B}}}{m^{1/4}n^{1/2}d^{1/4}} = \frac{\dfrac{\partial}{\partial u}\log(c'^3dm^3n)}{8\,m^{1/4}n^{1/2}d^{1/4}}, \\
i_5 &= \frac{\overset{(y)}{\mathfrak{C}}{}'' - \overset{(z)}{\mathfrak{C}}{}''}{m^{1/2}n^{1/4}c'^{1/4}} = \frac{\dfrac{\partial}{\partial v}\log(c'd^3mn^3)}{8\,m^{1/2}n^{1/4}c'^{1/4}}.
\end{aligned}
\tag{1}
$$

* Brussells paper, Section 10.

In consequence of the fact that i_1 is a constant, the fourth and fifth of equations (1) can be written

$$
(2)\qquad
\begin{aligned}
i_4 &= \frac{\dfrac{\partial}{\partial u}\log\left(m^{1/2}\,n^{1/4}\,c'^{1/4}\right)}{m^{1/4}\,n^{1/2}\,d^{1/4}},\\
i_5 &= \frac{\dfrac{\partial}{\partial v}\log\left(m^{1/4}\,n^{1/2}\,d^{1/4}\right)}{m^{1/2}\,n^{1/4}\,c'^{1/4}}.
\end{aligned}
$$

From (2) we find

$$
(3)\qquad
\begin{aligned}
\frac{\partial^2}{\partial u\,\partial v}\log\left(m^{1/2}\,n^{1/4}\,c'^{1/4}\right) &= i_4\,\frac{\partial}{\partial v}\left(m^{1/4}\,n^{1/2}\,d^{1/4}\right) = i_4\,i_5\,m^{3/4}\,n^{3/4}\,c'^{1/4}\,d^{1/4},\\
\frac{\partial^2}{\partial u\,\partial v}\log\left(m^{1/4}\,n^{1/2}\,d^{1/4}\right) &= i_5\,\frac{\partial}{\partial u}\left(m^{1/2}\,n^{1/4}\,c'^{1/4}\right) = i_4\,i_5\,m^{3/4}\,n^{3/4}\,c'^{1/4}\,d^{1/4}.
\end{aligned}
$$

Hence

$$
(4)\qquad \frac{\partial^2}{\partial u\,\partial v}\log\left(\frac{m c'}{n d}\right) = 0,
$$

and mc'/nd is the product of a function of u only by a function of v only. Hence, according to equations (I, 7) we can find a transformation of u and v which will make the transform of this expression identically equal to unity. Let us assume that this transformation has already been applied, so that

$$
(5)\qquad \frac{m c'}{n d} = 1.
$$

From the second and third of equations (1)

$$
(6)\qquad
\begin{aligned}
2f_u &= \frac{\partial}{\partial u}\log\left(c'^{3i_3}\,d^{i_3+1}\,m^{3i_2+3}\,n^{i_2}\right),\\
2f_v &= \frac{\partial}{\partial v}\log\left(c'^{i_2+1}\,d^{3i_2}\,m^{i_3}\,n^{3i_3+3}\right),
\end{aligned}
$$

whence

$$
(7)\qquad \frac{\partial^2}{\partial u\,\partial v}\log\left\{\frac{c'^{3i_3-i_2-1}\,m^{3i_2-i_3+3}}{d^{3i_2-i_3-1}\,n^{3i_3-i_2+3}}\right\} = 0,
$$

and

$$
\frac{c'^{3i_3-i_2-1}\,m^{3i_2-i_3+3}}{d^{3i_2-i_3-1}\,n^{3i_3-i_2+3}}
$$

is the product of a function of u only by a function of v only. From equations (I,7) we see that it is possible to find a transformation of y

and z only (under which (5) is preserved) which will make the transform of the expression in the bracket in (7) identically equal to unity. Let us assume that this transformation has already been applied, so that

$$\frac{c'^{\,3i_2-i_3-1}\, m^{3i_2-i_3+3}}{d^{3i_3-i_2-1}\, n^{3i_3-i_2+3}} = 1 . \tag{8}$$

From the first of equations (II,1) and equations (5) and (8)

$$\begin{aligned} \frac{c'}{n} = \frac{d}{m} &= i_1^{1/2}, \\ m^{i_2-i_3+1}\, n^{i_2-i_3-1} &= i_1^{-(i_2-i_3)/2}, \\ d^{i_3-i_2+1}\, n^{i_2-i_3-1} &= i_1^{1/2}. \end{aligned} \tag{9}$$

Substituting equations (9) in the last two of equations (1), we obtain

$$i_4 = \frac{\dfrac{\partial(mn)}{\partial u}}{2\, i_1^{1/8}(mn)^{3/2}}, \qquad i_5 = \frac{\dfrac{\partial(mn)}{\partial v}}{2\, i_1^{1/8}(mn)^{3/2}}, \tag{10}$$

whence

$$mn = \frac{1}{i_1^{1/4}(i_4 u + i_5 v)^2}. \tag{11}$$

From equations (9) and (11) we find

$$\begin{aligned} m &= \frac{(i_4 u + i_5 v)^{i_2-i_3-1}}{i_1^{(i_2-i_3+1)/8}}, \\ n &= \frac{i_1^{(i_2-i_3-1)/8}}{(i_4 u + i_5 v)^{i_2-i_3+1}}, \\ c' &= \frac{i_1^{(i_2-i_3+3)/8}}{(i_4 u + i_5 v)^{i_2-i_3+1}}, \\ d &= \frac{(i_4 u + i_5 v)^{i_2-i_3-1}}{i_1^{(i_2-i_3-3)/8}}, \end{aligned} \tag{12}$$

and from equations (6) and (12)

$$\begin{aligned} f_u &= \frac{-2\, i_4 (i_2 + i_3 + 1)}{i_4 u + i_5 v}, \\ f_v &= \frac{-2\, i_5 (i_2 + i_3 + 1)}{i_4 u + i_5 v}. \end{aligned} \tag{13}$$

Then from the integrability conditions (I,2) it follows that, if $W \neq 0$ (i. e., if $i_1 \neq 1$), the coefficients of the differential equations (D) can be written

$$
(14) \qquad
\begin{aligned}
m &= m_0\,(i_4 u + i_5 v)^{j-2}, & n &= n_0\,(i_4 u + i_5 v)^{-j},\\
a &= a_0\,(i_4 u + i_5 v)^{-2}, & b &= b_0\,(i_4 u + i_5 v)^{j-3},\\
c &= c_0\,(i_4 u + i_5 v)^{-1}, & d &= d_0\,(i_4 u + i_5 v)^{j-2},\\
a' &= a_0'\,(i_4 u + i_5 v)^{-j-1}, & b' &= b_0'\,(i_4 u + i_5 v)^{-2},\\
c' &= c_0'\,(i_4 u + i_5 v)^{-j}, & d' &= d_0'\,(i_4 u + i_5 v)^{-1},
\end{aligned}
$$

where the letters with subscripts 0 represent constants, and where

$$
(15) \qquad j = i_2 - i_3 + 1.
$$

Substituting equations (12) and (14) in (I, 2), we obtain

$$
(16) \qquad
\begin{aligned}
b_0 &= \frac{i_5\,(i_2 + 3\,i_3 + 3)}{i_1^{(i_2 - i_3 - 3)/8}},\\
a_0' &= \frac{i_4\,(3\,i_2 + i_3 + 3)}{i_1^{(i_3 - i_2 - 3)/8}},\\
i_2 + i_3 + 1 &= \frac{i_1^{-1/4} - i_1^{3/4}}{2\,i_4\,i_5},\\
(i_2 - i_3 - 1)(3\,i_2 + i_3)\,i_4^2 - i_1^{1/2}(i_2 - i_3 - 2)(i_2 + 3\,i_3 + 3)\,i_5^2 &= a_0 + i_1^{1/2}\,b_0'.\\
(i_2 - i_3 + 2)(3\,i_2 + i_3 + 3)\,i_4^2 - i_1^{-1/2}(i_2 - i_3 + 1)(i_2 + 3\,i_3)\,i_5^2 &= a_0 + i_1^{-1/2}\,b_0',\\
i_1^{3/4}(3\,i_2 + i_3 + 3)\,i_4 - i_1^{-1/4}(3\,i_2 + i_3 - 1)\,i_4 &= 2\,i_5\,a_0.\\
i_1^{3/4}(i_2 + 3\,i_3 + 3)\,i_5 - i_1^{-1/4}(i_2 + 3\,i_3 - 1)\,i_5 &= 2\,i_4\,b_0'.
\end{aligned}
$$

From the fourth and fifth of (16) we find

$$
(17) \qquad
\begin{aligned}
(i_1 - 1)\,a_0 &= [i_1(i_2 - i_3 + 2)(3\,i_2 + i_3 + 3) - (i_2 - i_3 - 1)(3\,i_2 + i_3)]\,i_4^2\\
&\quad - i_1^{1/2}(12\,i_3 + 6)\,i_5^2,\\
(i_1 - 1)\,b_0' &= -\,i_1^{1/2}(12\,i_2 + 6)\,i_4^2 - [i_1(i_2 - i_3 - 2)(i_2 + 3\,i_3 + 3)\\
&\quad - (i_2 - i_3 + 1)(i_2 + 3\,i_3)]\,i_5^2.
\end{aligned}
$$

But from the third, sixth, and seventh of (16) we have

$$
(18) \qquad
\begin{aligned}
(i_1 - 1)\,a_0 &= -\,(i_2 + i_3 + 1)[i_1(3\,i_2 + i_3 + 3) - (3\,i_2 + i_3 - 1)]\,i_4^2,\\
(i_1 - 1)\,b_0' &= -\,(i_2 + i_3 + 1)[i_1(i_2 + 3\,i_3 + 3) - (i_2 + 3\,i_3 - 1)]\,i_5^2,
\end{aligned}
$$

and from (17) and (18)

$$
(19) \qquad \begin{aligned}
&[i_1(2i_2+3)(3i_2+i_3+3)-(6i_2^2+2i_2i_3-i_2-i_3-1)]i_4^2 \\
&\qquad\qquad -i_1^{1/2}(12i_3+6)i_5^2 = 0, \\
&-i_1^{1/2}(12i_2+6)i_4^2+[i_1(2i_3+3)(i_2+3i_3+3) \\
&\qquad\qquad -(2i_2i_3+6i_3^2-i_2-i_3-1)]i_5^2 = 0.
\end{aligned}
$$

It can now be easily seen that if i_1, i_2, i_3, i_4, and i_5 are constants satisfying the third of equations (16) and equations (19) and if i_1, i_4, and i_5 are all different from zero and i_1 is different from 1, a congruence exists having the absolute invariants i_1, i_2, i_3, i_4, and i_5, as defined by equations (1), equal to the specified constants. If, further, a definite choice be made of the values of the fractional powers of i_1 in equations (12) and the first and second of equations (16), the congruence is uniquely determined except for projective transformations. For under these conditions equations (17) determine a_0 and b_0', which satisfy also equations (18) and the last four of equations (16). The first two of equations (16) determine a_0' and b_0. Then a, a', b, and b' are determined by equations (14), and c, c', d, d', m, and n are determined by equations (12) and (13). Then, according to the general theory, the congruence is determined except for projective transformations.

Since neither i_4 nor i_5 is zero, equations (19) imply

$$
(20) \qquad \begin{aligned}
&i_2^2(3i_2+i_3+3)(i_2+3i_3+3)(2i_2+3)(2i_3+3)-i_1[24i_2^3i_3+80i_2^2i_3^2 \\
&\quad +24i_2i_3^3+12i_2^3+116i_2^2i_3+116i_2i_3^2+12i_3^3+30i_2^2+140i_2i_3+30i_3^2 \\
&\quad +36i_2+36i_3+18]+[12i_2^3i_3+40i_2^2i_3^2+12i_2i_3^3-6i_2^3-10i_2^2i_3 \\
&\quad -10i_2i_3^2-6i_3^3-5i_2^2-2i_2i_3-5i_3^2+2i_2+2i_3+1] = 0.
\end{aligned}
$$

2. The differential equations of the two sheets of the focal surface are, by equations (I, 8) and (I,9)

$$
(21) \qquad \begin{aligned}
m_0y_{uu}-d_0y_{vv} &= \frac{a_0m_0}{(i_4u+i_5v)^2}y+\frac{c_0m_0}{(i_4u+i_5v)}y_u+\frac{b_0-i_5(j-2)d_0}{(i_4u+i_5v)}y_v, \\
y_{uv} &= \frac{m_0n_0}{(i_4u+i_5v)^2}y+\frac{i_4(j-2)}{(i_4u+i_5v)}y_v,
\end{aligned}
$$

and

$$
(22) \qquad \begin{aligned}
-c_0'z_{uu}+n_0z_{vv} &= \frac{b_0'n_0}{(i_4u+i_5v)^2}z+\frac{a_0'+i_4jc_0'}{(i_4u+i_5v)}z_u+\frac{d_0'n_0}{(i_4u+i_5v)}z_v, \\
z_{uv} &= \frac{m_0n_0}{(i_4u+i_5v)^2}z-\frac{i_5j}{(i_4u+i_5v)}z_u.
\end{aligned}
$$

In order to obtain a system of differential equations representing S_y referred to its asymptotic curves, we make the transformation

$$\bar{u} = \sqrt{-d_0}\, u + \sqrt{m_0}\, v, \qquad \bar{v} = \sqrt{-d_0}\, u - \sqrt{m_0}\, v,^{*}$$

under which equations (21) become

$$
(23)\qquad
\begin{aligned}
y_{\bar u \bar u} &= \frac{2 n_0 \sqrt{-d_0 m_0} - a_0}{4 d_0 \nu}\, y \\
&\quad - \frac{c_0 m_0 \sqrt{-d_0} + \sqrt{m_0}\,(b_0 - i_5 (j-2)\, d_0) - 2\, i_4 (j-2)\, m_0 \sqrt{-d_0}}{4 d_0 m_0 \nu}\, y_{\bar u} \\
&\quad - \frac{c_0 m_0 \sqrt{-d_0} + \sqrt{m_0}\,(b_0 - i_5 (j-2)\, d_0) + 2\, i_4 (j-2)\, m_0 \sqrt{-d_0}}{4 d_0 m_0 \nu}\, y_{\bar v}, \\
y_{\bar v \bar v} &= \frac{-2 n_0 \sqrt{-d_0 m_0} - a_0}{4 d_0 \nu}\, y \\
&\quad - \frac{c_0 m_0 \sqrt{-d_0} + \sqrt{m_0}\,(b_0 - i_5 (j-2)\, d_0) + 2\, i_4 (j-2)\, m_0 \sqrt{-d_0}}{4 d_0 m_0 \nu}\, y_{\bar u} \\
&\quad - \frac{c_0 m_0 \sqrt{-d_0} + \sqrt{m_0}\,(b_0 - i_5 (j-2)\, d_0) - 2\, i_4 (j-2)\, m_0 \sqrt{-d_0}}{4 d_0 m_0 \nu}\, y_{\bar v},
\end{aligned}
$$

where

$$(24)\qquad \nu = i_4 u + i_5 v = \frac{(i_4 \sqrt{m_0} + i_5 \sqrt{-d_0})\, \bar u + (i_4 \sqrt{m_0} - i_5 \sqrt{-d_0})\, \bar v}{2 \sqrt{-m_0 d_0}}.$$

If we write equations (23), for brevity, in the form

$$
(25)\qquad
\begin{aligned}
y_{\bar u \bar u} + 2\alpha\, y_{\bar u} + 2\beta\, y_{\bar v} + \gamma\, y &= 0, \\
y_{\bar v \bar v} + 2\alpha'\, y_{\bar u} + 2\beta'\, y_{\bar v} + \gamma'\, y &= 0,
\end{aligned}
$$

we see that the fundamental seminvariants a', b, f, g† are of degrees -1, -1, -2, -2 respectively in ν. Hence the relative invariants a', b, h, k are of degrees -1, -1, -4, -4 respectively. Since they are of weights $(-1, 2)$, $(2, -1)$, $(6, -2)$, $(-2, 6)$ respectively, the exponents

* Wilczynski, *Projective differential geometry of curved surfaces*, first memoir, these Transactions, vol. 8 (1907), pp. 233–260, equation (22); this memoir is cited hereafter as *Curved surfaces*.

† *Curved surfaces*, equation (38).

p, q, r, s in an absolute invariant of the form $a'^p b^q h^r k^s$ must satisfy the relations

$$(26)\qquad \begin{aligned} -p+2q+6r-2s &= 0, \\ 2p-\ q-2r+6s &= 0. \end{aligned}$$

Hence

$$(27)\qquad p+q+4r+4s = 0:$$

i. e., any absolute invariant of this type is constant.

From the invariants a', b, h, k, four others,

$$(28)\qquad A = a'b^2, \qquad B = a'^2 b, \qquad H = a'h, \qquad K = bk,$$

are derived, which have weights (3,0), (0,3), (5,0), (0,5) and degrees -3, -3, -5, -5 respectively. From these, all others can be obtained by means of the operations

$$U = \alpha' \frac{\partial}{\partial u}, \qquad V = \beta \frac{\partial}{\partial v}$$

and the Wronskian operation*; the U operation is applied only to invariants of zero u-weight, and adds 2 to the v-weight and -2 to the degree; the V operation is applied only to invariants of zero v-weight, and adds 2 to the u-weight and -2 to the degree. Since for each of the invariants thus far mentioned the sum of the two weights and the degree is zero, any invariant obtained from them by means of the U and V operations must have the same property. Hence, if the weights are both zero, the degree must be zero. Since the Wronskian operation is essentially partial differentiation of an absolute invariant, it can, under our hypothesis, yield only invariants which are identically zero. Hence all the absolute invariants of the y sheet of the focal surface are constants; the same proposition can evidently be proved in regard to the z sheet. It is also readily seen that both sheets are of the type discussed by Wilczynski in his paper *On a certain class of self-projective surfaces.*†

By successive differentiation of equations (21) and elimination of all terms involving differentiation with respect to v, we obtain for the curves $v =$ constant on S_y a differential equation of the form

$$(29)\qquad \begin{aligned} p_0\, y_{uuuu} + \frac{4p_1}{(i_4 u + i_5 v)}\, y_{uuu} + \frac{6p_2}{(i_4 u + i_5 v)^2}\, y_{uu} \\ + \frac{4p_3}{(i_4 u + i_5 v)^3}\, y_u + \frac{p_4 y}{(i_4 u + i_5 v)^4} = 0. \end{aligned}$$

* *Curved surfaces,* Section 7.

† These Transactions, vol. 14 (1913), pp. 421-443.

where p_0, p_1, p_2, p_3, and p_4 are constants. Since the absolute invariants of this equation are constants (independent of both u and v), the curves $v =$ constant on S_y are anharmonic curves, all projective to one another. In the same manner each of the families of curves $u =$ constant and $v =$ constant on S_y and S_z can be shown to consist of projectively equivalent anharmonic curves.*

If we write the differential equations of the 1st and (—1)st Laplace transforms according to formulas (I, 10), (I, 11), and (I, 12), we find that they have constant absolute invariants, $\overset{(y)}{\mathfrak{B}} \neq \overset{(z)}{\mathfrak{B}}$, and $\overset{(y)}{\mathfrak{C}}{}'' \neq \overset{(z)}{\mathfrak{C}}{}''$, but they are not projective to the original congruence.

3. In the special case, $i_1 = 1$, excluded from the above discussion subsequent to equation (13), the sixth and seventh of the integrability conditions can not be solved for a and b', and hence equations (14) are not valid. In this case equations (12) still hold (with $i_1 = 1$), but equations (13), together with the assumption that $f_{uv} = 0$, become

$$f_u = f_v = 0, \tag{30}$$

whence

$$i_2 + i_3 + 1 = 0. \tag{31}$$

The integrability conditions then take the form

$$\begin{aligned} c - d' = 0, \qquad b = -d_v, \qquad a' &= -c'_u, \\ m_{uu} + d_{vv} &= m a + d b', \\ n_{vv} + c'_{uu} &= c' a + n b', \\ 2 m_u n + m n_u &= a_v + a' d, \\ m_v n + 2 m n_v &= b_u + b c'. \end{aligned} \tag{32}$$

Substituting equations (12) in the last four of (32), we obtain

$$\begin{aligned} (i_2 - i_3 - 1)(i_2 - i_3 - 2)(i_4^2 + i_5^2) &= (a + b')(i_4 u + i_5 v)^2, \\ (i_2 - i_3 + 1)(i_2 - i_3 + 2)(i_4^2 + i_5^2) &= (a + b')(i_4 u + i_5 v)^2, \\ a_v = \frac{-4 i_4}{(i_4 u + i_5 v)^3}, \qquad b'_u &= \frac{-4 i_5}{(i_4 u + i_5 v)^3}. \end{aligned} \tag{33}$$

* Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, pp. 243, 279 (cited hereafter as *Projective Differential Geometry*).

From the last two of equations (33)

$$a = \frac{2 i_4}{i_5 (i_4 u + i_5 v)^2} + \text{(a function of } u \text{ only)},$$
$$b' = \frac{2 i_5}{i_4 (i_4 u + i_5 v)^2} + \text{(a function of } v \text{ only)}.$$

Substituting these expressions in the first two of equations (33) we see that

$$\begin{aligned} a &= \frac{2 i_4}{i_5 (i_4 u + i_5 v)^2} + k, \\ b' &= \frac{2 i_5}{i_4 (i_4 u + i_5 v)^2} - k, \end{aligned} \tag{34}$$

where k is a constant independent of the invariants, and that

$$\begin{aligned} (i_2 - i_3 - 1)(i_2 - i_3 - 2)(i_4^2 + i_5^2) &= (i_2 - i_3 + 1)(i_2 - i_3 + 2)(i_4^2 + i_5^2) \\ &= \frac{2(i_4^2 + i_5^2)}{i_4 i_5}. \end{aligned} \tag{35}$$

From the first two members of this equation it follows that

$$(i_2 - i_3)(i_4^2 + i_5^2) = 0.$$

From equations (12) we find

$$\begin{aligned} m &= d = (i_4 u + i_5 v)^{i_2 - i_3 - 1}, \\ n &= c' = (i_4 u + i_5 v)^{-i_2 + i_3 - 1}. \end{aligned} \tag{36}$$

As in (II, 1) it can be shown that if $i_1 (= 1)$ and i_2, i_3, i_4, i_5, and k be given as constants satisfying equations (35) and if neither i_4 nor i_5 is zero, the congruence is determined except for projective transformations.

As before, we see that the successive Laplace transforms are all projectively distinct.

III. Case 2, $\overset{(y)}{\mathfrak{B}} = \overset{(z)}{\mathfrak{B}}$, $\overset{(y)}{\mathfrak{C}}{}'' = \overset{(z)}{\mathfrak{C}}{}''$

1. We shall use the absolute invariant i_1 defined in equations (II, 1) and the absolute invariants

$$
(1) \qquad
\begin{aligned}
j_4 &= \frac{\overset{(y)}{\mathfrak{B}}}{m^{1/4}\, n^{1/2}\, d^{1/4}}, \\
j_5 &= \frac{\overset{(z)}{\mathfrak{C}''}}{m^{1/2}\, n^{1/4}\, c'^{1/4}}.
\end{aligned}
$$

From equations (I, 6) and the hypotheses of this section

$$
(2) \qquad
\begin{aligned}
c'^3\, d m^3 n &= \varphi(v), \\
c'\, d^3 m n^3 &= \psi(u).
\end{aligned}
$$

Since, according to equations (I, 7), under the transformations (I, 4) and (I, 5),

$$
(3) \qquad
\begin{aligned}
\overline{c'^3\, d m^3 n} &= \beta_v^{-2} c'^3 d m^3 n, \\
\overline{c'\, d^3 m n^3} &= \alpha_u^{-2} c'\, d^3 m n^3,
\end{aligned}
$$

we can choose α and β so that $\overline{c'^3 d m^3 n} = \overline{c' d^3 m n^3} = 1$. Let us assume that this transformation has already been applied, i. e., that

$$
(4) \qquad c'^3 d m^3 n = c' d^3 m n^3 = 1.
$$

Hence, according to (I, 7) and (3), we can choose α and β so that

$$
(5) \qquad c' m = d n = 1.
$$

From equations (II, 1) and (5)

$$
(6) \qquad
\begin{aligned}
m &= i_1^{-1/2} n^{-1}, \\
c' &= i_1^{1/2} n, \\
d &= n^{-1}.
\end{aligned}
$$

From equations (I, 6), (1), and (6)

$$
(7) \qquad
\begin{aligned}
\frac{f_u}{4} + \frac{n_u}{2n} &= i_1^{-1/8} j_4, \\
\frac{f_v}{4} - \frac{n_v}{2n} &= i_i^{-1/8} j_5.
\end{aligned}
$$

Hence

$$
(8) \qquad f_{uv} = -2 \frac{\partial^2}{\partial u\, \partial v} \log n = 2 \frac{\partial^2}{\partial u\, \partial v} \log u,
$$

and

$$
(9) \qquad \begin{aligned} f_{uv} &= 0, \\ \frac{\partial^2}{\partial u\, \partial v} \log n &= 0. \end{aligned}
$$

Hence n is the product of a function of u only by a function of v only, and can be reduced to unity by a transformation of form (I, 5), which does not disturb equations (5). Let us assume that this has already been done. From the second of equations (I, 2), the definition of i_1, and the first of equations (9)

$$
(10) \qquad i_1 = 1.
$$

Then, since $n = 1$, equations (6) and (10) give

$$
(11) \qquad m = n = c' = d = 1.
$$

Hence from (7) and (11)

$$
(12) \qquad \begin{aligned} f_u &= 4j_4, \\ f_v &= 4j_5. \end{aligned}
$$

The integrability conditions can now be written

$$
(13) \qquad \begin{aligned} c &= 4j_4, \qquad & d &= 4j_5, \\ b &= -4j_5, \qquad & a' &= -4j_4, \\ W &= mn - c'd = 0, \\ a + b' &= 0, \\ a_v &= 0, \qquad & b'_u &= 0. \end{aligned}
$$

From the last three of equations (13)

$$
(14) \qquad a = k, \qquad b' = -k,
$$

where k is a constant independent of the invariants.

The differential equations of the congruence are therefore

$$
(15) \qquad \begin{aligned} y_v &= z, \qquad z_u = y, \\ y_{uu} &= ky - 4j_5 z + 4j_4 y_u + z_v, \\ z_{vv} &= -4j_4 y - kz + y_u + 4j_5 z_v. \end{aligned}
$$

Obviously the congruence is determined, except for projective transformations, by the (constant) values of $i_1 (= 1)$, j_4, j_5, and k.

2. The differential equations of the two sheets of the focal surface are

$$(16)\qquad \begin{aligned} y_{uu} - y_{vv} &= ky + 4j_4 y_u - 4j_5 y_v, \\ y_{uv} &= y. \end{aligned}$$

$$(17)\qquad \begin{aligned} z_{uu} - z_{vv} &= kz + 4j_4 z_u - 4j_5 z_v, \\ z_{uv} &= z. \end{aligned}$$

(See (I, 8) and (I, 9).) The two sheets are obviously projective to one another, with each point on one sheet corresponding to the point on the other which is given by the same pair of values of u and v.

It is easily seen that each of the surfaces S_y and S_z has constant absolute invariants*. The net of curves $u =$ constant and $v =$ constant on each surface is isothermally conjugate and has equal Laplace-Darboux invariants.†

As indicated in § 2, we find the differential equations of the four families of parametric curves to be

$$(18)\qquad y_{uuuu} - 4j_4 y_{uuu} - k y_{uu} + 4j_5 y_u - y = 0,$$

$$(19)\qquad y_{vvvv} - 4j_5 y_{vvv} + k y_{vv} + 4j_4 y_v - y = 0,$$

$$(20)\qquad z_{uuuu} - 4j_4 z_{uuu} - k z_{uu} + 4j_5 z_u - z = 0,$$

$$(21)\qquad z_{vvvv} - 4j_5 z_{vvv} + k z_{vv} + 4j_4 z_v - z = 0.$$

Since each of these differential equations has constant seminvariants, each of the four families of parametric curves consists of projectively equivalent anharmonic curves; in fact, the curves represented by equations (18) and (20) are all projectively equivalent; similarly the curves represented by equations (19) and (21) are all projectively equivalent. Furthermore, since the differential equations, for example, of any curve, $v = c_1$, of the family of curves $v =$ constant on S_y can be transformed into the differential equation of any other curve, $v = c_2$, of the same family, *without change of the independent variable* u,‡ it follows that any two curves of the family $v =$ constant on S_y are projective to one another, any pair of

* *Curved surfaces*, 7.

† Wilczynski, *General theory of congruences*, these Transactions, vol. 16 (1915), pp. 318–322.

‡ *Projective Differential Geometry*, pp. 239, 242.

corresponding points lying on the same curve of the family $u=$ constant on S_y. Similarly it can be seen that any two curves of the family $u=$ constant on S_y are projective to one another, any pair of corresponding points lying on the same curve of the family $v=$ constant on S_y. Evidently the other focal sheet, S_z, has the same property. Furthermore, since each developable surface of the congruence is completely determined by its cuspidal edge and since two developables are projective to one another if (and only if) their cuspidal edges are projective to one another, it follows that any two developables of the congruence belonging to one family are projective to one another, any pair of corresponding generators lying on the same developable of the other family.

A fundamental system of solutions of equations (15) is

$$\begin{aligned} y_i &= e^{e_i u + v/e_i}, \\ z_i &= \frac{1}{e_i} e^{e_i u + v/e_i} \qquad (i = 1, 2, 3, 4), \end{aligned} \tag{22}$$

where $e_1, \cdots, e_4$ are the roots (assumed to be distinct) of the equation

$$e^4 - 4j_4 e^3 - k e^2 + 4 j_5 e - 1 = 0. \tag{23}$$

The Plücker coördinates of the lines of the congruence are then

$$\omega_{ij} = \left(\frac{1}{e_j} - \frac{1}{e_i}\right) e^{(e_i + e_j)(u + v/e_i e_j)} \qquad (i, j = 1, \cdots, 4). \tag{24}$$

Hence the congruence belongs to the tetrahedral complex

$$\frac{\omega_{12}\,\omega_{34}}{(e_2-e_1)(e_4-e_3)} = \frac{\omega_{13}\,\omega_{42}}{(e_3-e_1)(e_2-e_4)} = \frac{\omega_{14}\,\omega_{23}}{(e_4-e_1)(e_3-e_2)}. \tag{25}$$

These two equations represent one and the same complex, since from the first, viz.

$$\frac{\omega_{12}\,\omega_{34}}{(e_2-e_1)(e_4-e_3)} = \frac{\omega_{13}\,\omega_{42}}{(e_3-e_1)(e_2-e_4)},$$

together with the fundamental identity

$$\omega_{12}\,\omega_{34} + \omega_{13}\,\omega_{42} + \omega_{14}\,\omega_{23} = 0,$$

it follows that

$$\frac{\omega_{14}\,\omega_{23}}{(e_4-e_1)(e_3-e_2)} = \frac{-\omega_{12}\,\omega_{34}-\omega_{13}\,\omega_{42}}{(e_4-e_1)(e_3-e_2)}$$

$$= \left[\frac{-(e_2-e_1)(e_4-e_3)-(e_3-e_1)(e_2-e_4)}{(e_4-e_1)(e_3-e_2)(e_2-e_1)(e_4-e_3)}\right]\omega_{12}\,\omega_{34}$$

$$= \frac{\omega_{12}\,\omega_{34}}{(e_2-e_1)(e_4-e_3)}\,.$$

In order to obtain another equation of the congruence we replace the subscript j in equation (24) by k and divide the resulting equation by equation (24), obtaining the homogeneous equation

$$\frac{\omega_{ij}}{\omega_{ik}} = \frac{(e_i-e_j)\,e_k}{(e_i-e_k)\,e_j}\,e^{(e_j-e_k)(u-v)/e_j e_k},$$

where i, j, and k are any three distinct numbers of the set 1, 2, 3, 4. If we take the logarithm of each side, set (i, j, k) equal in turn to (1, 2, 3), (2, 3, 4), and (3, 4, 1), and eliminate u and v from the three resulting equations, we obtain, as the equation of another complex to which our congruence belongs,

$$\frac{e_2(e_3-e_1)}{(e_2-e_3)}\log\left[\frac{(e_1-e_3)\,e_2\,\omega_{12}}{(e_1-e_2)\,e_3\,\omega_{13}}\right] + \frac{(e_4 e_1-e_2 e_3)}{(e_3-e_1)}\log\left[\frac{(e_4-e_2)\,e_3\,\omega_{23}}{(e_2-e_3)\,e_1\,\omega_{12}}\right]$$

$$+\frac{e_1(e_2-e_4)}{(e_4-e_1)}\log\left[\frac{(e_1-e_3)\,e_4\,\omega_{34}}{(e_3-e_1)\,e_1\,\omega_{23}}\right] = 0.$$

If the roots of equation (23) are not all distinct, equations (15) can still be easily solved, and it can be shown in each case that the congruence belongs to a quadratic complex. In each case the congruence belongs to a complex obtained from a tetrahedral complex by making some or all of the faces of the fundamental tetrahedron approach coincidence.

From equations (I, 11) and (I, 12) it is evident that the differential equations of the 1st and (-1)st Laplace transforms of the congruence (15), and hence of all the Laplace transforms, are identical with equations (15). Hence each Laplace transform is projective to the original congruence, corresponding lines being determined by the same pair of values of u and v.

3. Conversely, if a congruence is projective to its 1st and (-1)st Laplace transforms, corresponding lines being given by the same pair of values of u and v, the congruence is of the type discussed in this section.

This hypothesis implies that there exist functions λ and λ' of u only and μ and μ' of v only such that transformations (I,5) will convert the differential equations of the original congruence into those of its 1st Laplace transform if the functions λ, μ are used, and into those of its (-1)st Laplace transform if the functions λ', μ' are used. Hence, in Wilczynski's notation,

$$
(26) \qquad
\begin{aligned}
m_1 &= m\left(mn - \frac{\partial^2}{\partial u\,\partial v}\log m\right) &&= \frac{\mu m}{\lambda},\\
n_1 &= \frac{1}{m} &&= \frac{\lambda n}{\mu},\\
c_1' &= \frac{1}{d} &&= \frac{\lambda}{\mu}c',\\
d_1 &= d\left(c'd - \frac{\partial^2}{\partial u\,\partial v}\log d\right) &&= \frac{\mu d}{\lambda},
\end{aligned}
$$

$$
(27) \qquad
\begin{aligned}
m_{-1} &= \frac{1}{n} &&= \frac{\mu' m}{\lambda'},\\
n_{-1} &= n\left(mn - \frac{\partial^2}{\partial u\,\partial v}\log n\right) &&= \frac{\lambda' n}{\mu'},\\
c'_{-1} &= c'\left(c'd - \frac{\partial^2}{\partial u\,\partial v}\log c'\right) &&= \frac{\lambda' c'}{\mu'},\\
d_{-1} &= \frac{1}{c'} &&= \frac{\mu' d}{\lambda'}.
\end{aligned}
$$

From equations (26) and (27)

$$
(28) \qquad
\begin{aligned}
mn - \frac{\partial^2}{\partial u\,\partial v}\log m &= mn = \frac{\mu}{\lambda},\\
mn - \frac{\partial^2}{\partial u\,\partial v}\log n &= mn = \frac{\lambda'}{\mu'},\\
c'd - \frac{\partial^2}{\partial u\,\partial v}\log c' &= c'd = \frac{\lambda'}{\mu'},\\
c'd - \frac{\partial^2}{\partial u\,\partial v}\log d &= c'd = \frac{\mu}{\lambda}.
\end{aligned}
$$

Equations (28) imply that each of the coefficients m, n, c', d is the product of a function of u alone by a function of v alone. Hence it is possible by means of transformations of types (I,4) and (I,5) to reduce m

and n to unity; let us assume that this has already been done, and that equations (26), (27) and (28) apply to the coefficients in this form. Thus

$$\begin{aligned} m &= n = 1, \\ \lambda &= \mu, \\ \lambda' &= \mu'. \end{aligned} \tag{29}$$

Hence λ, μ, λ', and μ' are constants.

Next note that

$$\begin{aligned} c_1 &= f_u + \frac{\partial}{\partial u}\log\left(\frac{d}{m}\right) = f_u - \frac{2\lambda_u}{\lambda}, \\ d'_1 &= f_v + \frac{\partial}{\partial v}\log\left(\frac{d}{m}\right) = f_v - \frac{2\mu_v}{\mu}, \\ c_{-1} &= f_u + \frac{\partial}{\partial u}\log\left(\frac{c'}{n}\right) = f_u - \frac{2\lambda'_u}{\lambda'}, \\ d'_{-1} &= f_v + \frac{\partial}{\partial v}\log\left(\frac{c'}{n}\right) = f_v - \frac{2\mu'_v}{\mu'}. \end{aligned} \tag{30}$$

From (29), (30) and the fact that λ, μ, λ', and μ' are constants, it follows that c' and d are constants. From (28) and (29)

$$c'd = 1. \tag{31}$$

Hence, according to (I, 7) it is possible to reduce c' and d to unity without disturbing m and n. Let us assume that this has already been done, and that in equations (26), (27), (28), (29), (30), and (31)

$$m = n = c' = d = 1. \tag{32}$$

Since

$$\begin{aligned} a_1 &= a + f_{uu} = a, \\ b'_{-1} &= b' + f_{vv} = b', \end{aligned} \tag{33}$$

and since, from (29) and (31),

$$f_{uv} = mn - c'd = 0, \tag{34}$$

it follows that f_u and f_v are constants.

$$\begin{aligned} f_u &= 4j_4, \\ f_v &= 4j_5. \end{aligned} \tag{35}$$

From the first and second of the integrability conditions (I, 2)

$$(36) \qquad \begin{aligned} a' &= -f_u = -4j_4\,, \\ b &= -f_v = -4j_5\,. \end{aligned}$$

From the fourth, fifth, sixth, and seventh of the integrability conditions (I, 2)

$$(37) \qquad a = k\,, \qquad b' = -k\,,$$

where k is a constant.

We can now easily prove the stronger theorem that if a congruence Γ is projective to its 1st Laplace transform Γ_1, each line of the original congruence corresponding to the line of the Laplace transform which touches a focal sheet at the same point, then the congruence has constant absolute invariants. For, in the first place, since the Laplace transform is projectively defined, any congruence projective to the given one is projective to the first Laplace transform of the former. Hence Γ_1 is projective to the second Laplace transform, Γ_2, of the original congruence. Thus Γ_1 is projective to its first Laplace transform Γ_2 and to its (-1)st Laplace transform Γ, with correspondence as described above. Hence, as we have just shown, Γ must be projective to its 1st and (-1)st Laplace transforms, and therefore, by the theorem just proved, Γ has constant absolute invariants, with $\overset{(y)}{\mathfrak{B}} = \overset{(z)}{\mathfrak{B}}$, and $\overset{(y)}{\mathfrak{C}}{}'' = \overset{(z)}{\mathfrak{C}}{}''$.

IV. Case 3, $\overset{(y)}{\mathfrak{B}} = \overset{(z)}{\mathfrak{B}}$, $\overset{(y)}{\mathfrak{C}}{}'' \neq \overset{(z)}{\mathfrak{C}}{}''$

1. It is convenient to use the absolute invariants

$$(1) \qquad \begin{aligned} i_1 &= \frac{c'd}{mn}\,, \\ j_4 &= \frac{\overset{(y)}{\mathfrak{B}}}{m^{1/4}\,n^{1/2}\,d^{1/4}}\,, \\ j_5 &= \frac{\overset{(z)}{\mathfrak{C}}{}''}{m^{1/2}\,n^{1/4}\,c'^{1/4}}\,, \\ j_6 &= \frac{\overset{(y)}{\mathfrak{C}}{}''}{m^{1/2}\,n^{1/4}\,c'^{1/4}}\,. \end{aligned}$$

Since

$$(2) \qquad 8\,(\overset{(z)}{\mathfrak{B}} - \overset{(y)}{\mathfrak{B}}) = \frac{\partial}{\partial u} \log\,(c'^8 dm^3 n) = 0\,.$$

we can, by a suitable transformation of v, make

$$c'^3 d m^3 n = \frac{i_1}{(j_6 - j_5)^8 r^8}. \tag{3}$$

Let us assume that this has already been done. Then from equations (1) and (3)

$$\frac{\partial}{\partial r} \log (c' d^3 m n^3) = 8\, m^{1/2} n^{1/4} c'^{1/4} (j_6 - j_5) = \frac{8}{r}. \tag{4}$$

Hence, by a transformation of u, we can make

$$c' d^3 m n^3 = r^8 \tag{5}$$

without altering equation (3). Let us assume that this has already been done. From (3), (5), and the first of (1)

$$\begin{aligned} c' d &= \frac{i_1^{5/8}}{j_6 - j_5}, \\ m n &= \frac{i_1^{-3/8}}{j_6 - j_5}, \\ c' m &= \frac{i_1^{3/8}}{(j_6 - j_5)^3 r^4}, \\ d n &= i_1^{-1/8} (j_6 - j_5) r^4. \end{aligned} \tag{6}$$

From equations (1) and (6)

$$\begin{aligned} \overset{(y)}{\mathfrak{B}} = \overset{(z)}{\mathfrak{B}} &= i_1^{-1/8} j_4 r, \\ \overset{(y)}{\mathfrak{C}}{}'' &= \frac{j_5}{(j_6 - j_5) r}, \\ \overset{(z)}{\mathfrak{C}}{}'' &= \frac{j_6}{(j_6 - j_5) r}. \end{aligned} \tag{7}$$

From equations (1) and (7)

$$\frac{\partial^2}{\partial u\, \partial r} \log (d m) = 2 \frac{\partial \overset{(y)}{\mathfrak{C}}{}''}{\partial u} - 2 \frac{\partial \overset{(y)}{\mathfrak{B}}}{\partial r} = \frac{-2 j_4}{i_1^{1/8}}. \tag{8}$$

Hence by a transformation of y and z, which leaves equations (6) and (7) unchanged, we can obtain

$$d m = r^4 e^{-2kur}, \tag{9}$$

where

$$k = i_1^{-1/8} j_4. \tag{10}$$

From equations (6) and (9)

$$\begin{aligned} m &= \frac{e^{-kuv}}{i_1^{1/8}(j_6 - j_5)}, \\ n &= i_1^{-1/4} e^{kuv}, \\ c' &= \frac{i_1^{1/2} e^{kuv}}{(j_6 - j_5)^2 v^4}, \\ d &= i_1^{1/8}(j_6 - j_5) v^4 e^{-kuv}. \end{aligned} \tag{11}$$

From equations (I, 6), (7) and (11),

$$\begin{aligned} f_u &= 2kv, \\ f_v &= 2ku + \frac{6j_5 - 2j_6}{(j_6 - j_5)v}. \end{aligned} \tag{12}$$

The integrability conditions can be written

$$\begin{aligned} b &= i_1^{1/8}[k(j_5 - j_6)uv^4 - 2(j_6 + j_5)v^3]e^{-kuv}, \\ a' &= \frac{-3i_1^{1/2} k e^{kuv}}{(j_6 - j_5)^2 v^3}, \\ i_1^{-1/4} - i_1^{3/4} &= 2j_4(j_6 - j_5), \\ 3k^2v^2 + 6i_1^{1/4}(j_6^2 - j_5^2)v^2 + 2i_1^{1/4}k(j_6 - j_5)&(j_6 - 3j_5)uv^3 - i_1^{1/4}k^2(j_6 - j_5)^2u^2v^4 \\ &= a + i_1^{1/4}(j_6 - j_5)^2 v^4 b', \\ -k^2(j_6 - j_5)^2u^2v^4 + 3i_1^{3/4}k^2v^2 &+ 2k(j_6 - j_5)(j_6 - 3j_5)uv^3 \\ &= i_1^{3/4} a + (j_6 - j_5)^2 v^4 b', \\ a_v &= \frac{3(i_1^{5/8} - i_1^{-3/8})kv}{(j_6 - j_5)} = -6k^2 v, \\ b'_u &= \frac{(i_1 - 1)ku}{i_1^{3/8}(j_6 - j_5)} + \frac{2[i_1^{5/8}(j_6 + j_5)^2 + i_1^{-3/8}(j_6 - 3j_5)]}{(j_6 - j_5)^2 v}. \end{aligned} \tag{13}$$

If i_1 were equal to 1, j_4, and hence k, would vanish. It would then follow from (11) that $\overset{(y)}{\mathfrak{C}}{}'' \equiv \overset{(z)}{\mathfrak{C}}{}''$, which is contrary to hypothesis. Hence $i_1 \neq 1$ and the fourth and fifth of equations (13) give

$$
(14)\qquad
\begin{aligned}
a &= 3k^2 v^2 + \frac{6\, i_1^{1/4}(j_6^2 - j_5^2)v^2}{(1 - i_1)},\\
b' &= \frac{2k(j_6 - 3j_5)u}{(j_6 - j_5)v} - \frac{6\, i_1 (j_6 + j_5)}{(1 - i_1)\,(j_6 - j_5)v^2} - k^2 u^2.
\end{aligned}
$$

From the sixth of equations (13) and the first of (14)

$$
(15)\qquad j_6^2 - j_5^2 = i_1^{-1/4} k^2 (i_1 - 1).
$$

From the seventh of (13) and the second of (14)

$$
(16)\qquad i_1^{3/8} k(j_6 - 3j_5)(j_6 - j_5) - i_1 (j_6 + j_5) + (3j_5 - j_6) = 0.
$$

From the third of (13) and (15)

$$
(17)\qquad
\begin{aligned}
j_5 &= \frac{-j_4^3}{i_1^{1/4}} - \frac{(1 - i_1)}{4\, i_1^{1/4} j_4},\\
j_6 &= \frac{-j_4^3}{i_1^{1/4}} + \frac{(1 - i_1)}{4\, i_1^{1/4} j_4}.
\end{aligned}
$$

From (16) and (17)

$$
(18)\qquad 2j_4^4 + 1 + i_1 = 0.
$$

From (17) and (18)

$$
(19)\qquad
\begin{aligned}
j_5 &= \frac{1 + 3i_1}{4\, i_1^{1/4} j_4},\\
j_6 &= \frac{3 + i_1}{4\, i_1^{1/4} j_4}.
\end{aligned}
$$

Substituting equations (19) in (11), (12), (13), and (14), we find

$$
\begin{aligned}
m &= \frac{2\, i_1^{1/8} j_4 e^{-kuv}}{(1 - i_1)},\\
n &= i_1^{-1/4} e^{kuv},\\
a &= -3\, i_1^{-1/4} j_4^2 v^2,\\
b &= \left[\frac{(i_1 - 1)}{2\, i_1^{1/4}} u v^4 + \frac{4 j_4^3 v^3}{i_1^{1/8}}\right] e^{-kuv},
\end{aligned}
$$

$$
(20)\qquad
\begin{aligned}
c &= 2 i_1^{-1/8} j_4 v, \\
d &= \frac{(1-i_1) v^4 e^{-kuv}}{2 i_1^{1/8} j_4}, \\
a' &= \frac{-12 i_1^{7/8} j_4^3 e^{kuv}}{(1-i_1)^2 v^3}, \\
b' &= \frac{-12(1+i_1)}{(1-i_1)^2 v^2} - \frac{8 i_1^{7/8} j_4 u}{(1-i_1) v} - \frac{j_4^2 u^2}{i_1^{1/4}}, \\
c' &= \frac{4 i_1 j_4^2 e^{kuv}}{(1-i_1)^2 v^4}, \\
d' &= \frac{2 j_4 u}{i_1^{1/8}} + \frac{8 i_1}{(1-i_1) v}.
\end{aligned}
$$

Evidently, if the absolute invariant j_4 be given equal to a constant different from zero and from $\sqrt[4]{-1}$ and if definite values be assigned to the fractional powers of i_1, the congruence is determined except for projective transformations; for i_1, j_5, j_6, and k are determined by equations (18), (17), and (10), and the coefficients of the differential equations are determined by equations (20).

2. The differential equations of the two sheets of the focal surface are, according to equations (I, 8) and (I, 9),

$$
(21)\qquad
\begin{aligned}
y_{uu} - \frac{(1-i_1)^2 v^4}{4 i_1^{1/4} j_4^2} y_{vv} &= -3 i_1^{-1/4} j_4^2 v^2 y + 2 i_1^{-1/8} j_4 v y_u + \frac{2(1-i_1) j_4^2 v^3}{i_1^{1/4}} y_v, \\
y_{uv} &= \frac{2 j_4}{i_1^{1/8}(1-i_1)} y - i_1^{-1/8} j_4 v y_v;
\end{aligned}
$$

$$
(22)\qquad
\begin{aligned}
\frac{4 i_1^{5/4} j_4^2}{(1-i_1)^2 v^4} z_{uu} - z_{vv} &= \left[\frac{12(1+i_1)}{(1-i_1)^2 v^2} + \frac{8 i_1^{7/8} j_4 u}{(1-i_1) v} + \frac{j_4^2 u^2}{i_1^{1/4}}\right] z \\
&\quad + \frac{16 i_1^{9/8} j_4^3}{(1-i_1)^2 v^8} z_u - \left[\frac{2 j_4 u}{i_1^{1/8}} + \frac{8 i_1}{(1-i_1) v}\right] z_v \\
z_{uv} &= \frac{2 j_4}{i_1^{1/8}(1-i_1)} z + i_1^{-1/8} j_4 u z_u.
\end{aligned}
$$

From equations (21) we find the equations of the parametric curves on the surface S_y to be

$$y_{uuuu} = 0, \tag{23}$$

$$\begin{aligned} y_{vvvv} + \frac{4(1-3i_1)}{(1-i_1)v} y_{vvv} - \frac{12 i_1(1-3i_1)}{(1-i_1)^2 v^2} y_{vv} + \frac{8 i_1 (1-3i_1)(1+i_1)}{(1-i_1)^3 v^3} y_v \\ - \frac{4 i_1 (1-3i_1)(1+i_1)}{(1-i_1)^4 v^4} y = 0. \end{aligned} \tag{24}$$

Evidently the curves $v =$ constant on S_y are cubics (obviously all projective to one another).

In order to simplify the computation and the resulting form of the differential equations of the parametric curves on the surface S_z we make the transformation

$$z = e^{kuv} \bar{z},$$

under which equations (22) become

$$\begin{aligned} \frac{4 i_1^{5/4} j_4^2}{(1-i_1)^2 v^4} \bar{z}_{uu} - \bar{z}_{vv} &= \frac{6(1+i_1)(2-i_1)}{(1-i_1)^2 v^2} \bar{z} + \frac{8 i_1^{9/8} j_4^3}{(1-i_1)^2 v^3} \bar{z}_u \\ &\quad - \frac{8 i_1}{(1-i_1) v} \bar{z}_v \\ \bar{z}_{uv} &= \frac{2 j_4}{i_1^{1/8}(1-i_1)} \bar{z} - \frac{j_4 v}{i_1^{1/8}} \bar{z}_v. \end{aligned} \tag{25}$$

The differential equations of the parametric curves on S_z are

$$\begin{aligned} \bar{z}_{uuuu} + \frac{6(1-i_1) j_4^2 v^2}{i_1^{5/4}} \bar{z}_{uu} - \frac{2(1-i_1)(3+2i_1) v^3}{i_1^{11/8} j_4} \bar{z}_u \\ - \frac{(1-i_1)(7+3i_1) v^4}{2 i_1^{3/2}} \bar{z} = 0; \end{aligned} \tag{26}$$

$$\begin{aligned} \bar{z}_{vvvv} + \frac{8(1-2i_1)}{(1-i_1) v} \bar{z}_{vvv} + \frac{12(2-5i_1+5i_1^2)}{(1-i_1)^2 v^2} \bar{z}_{vv} \\ + \frac{4(12-17i_1+8i_1^2-11i_1^3)}{(1-i_1)^3 v^3} \bar{z}_v \\ + \frac{4(1+i_1)(6-20i_1+19i_1^2-3i_1^3)}{(1+i_1)^4 v^4} \bar{z} = 0. \end{aligned} \tag{27}$$

It is easily seen that each of the differential equations (23), (24), (26), and (27) has constant absolute invariants, and therefore represents a family

of projectively equivalent anharmonic curves. Hence all the developable surfaces of each family are projectively equivalent. Furthermore, equations (24) and (27) have seminvariants independent of u; hence, as in III, since the differential equation of any curve $u =$ constant on either sheet of the focal surface can be transformed into the differential equation of any other curve $u =$ constant on the same sheet by means of a transformation of the type $y = \lambda \bar{y}$ or $z = \lambda \bar{z}$, where λ is a function of u und v, it follows that, in the projective correspondence between any two curves $u =$ constant on the same sheet of the focal surface, any pair of corresponding points lie on the same curve $v =$ constant on the focal sheet in question. Similarly, the curves $v =$ constant on S_y are projectively equivalent, corresponding points lying on the same curve $u =$ constant on S_y.

By means of formulas (I,10), (I,11), and (I,12) it can easily be shown that both the 1st and the (-1)st Laplace transforms of our congruence are of the same type as the original congruence (i. e., have constant absolute invariants), with $\overset{(y)}{\mathfrak{B}} \equiv \overset{(z)}{\mathfrak{B}}$, $\overset{(y)}{\mathfrak{C}}{}'' \not\equiv \overset{(z)}{\mathfrak{C}}{}''$ but are not projective to it. If $i_1 = 3$ the sheet $S_{y^{(1)}}$ of the focal surface of the 1st Laplace transform degenerates into a curve, and if $i_1 = 1/3$ the sheet $S_{z^{(-1)}}$ of the focal surface of the (-1)st Laplace transform degenerates into a curve.

The above discussion can be applied to congruences having $\overset{(y)}{\mathfrak{B}} \not\equiv \overset{(z)}{\mathfrak{B}}$, $\overset{(y)}{\mathfrak{C}}{}'' \equiv \overset{(z)}{\mathfrak{C}}{}''$ by interchanging u and v, y and z, and making other corresponding changes.

V. Summary of results

It is assumed throughout this thesis that the absolute invariants of the congruences discussed are constants and that m, n, c', and d are all different from zero. The congruences divide themselves into three main types in which, respectively, neither, both, or only one of the relative invariants $(\overset{(z)}{\mathfrak{B}} - \overset{(y)}{\mathfrak{B}})$, $(\overset{(y)}{\mathfrak{C}}{}'' - \overset{(z)}{\mathfrak{C}}{}'') = 0$.

In the first case, that in which neither $(\overset{(z)}{\mathfrak{B}} - \overset{(y)}{\mathfrak{B}})$ nor $(\overset{(y)}{\mathfrak{C}}{}'' - \overset{(z)}{\mathfrak{C}}{}'') = 0$, if the congruences are not W congruences the differential equations can be reduced by a suitable transformation of the variables to a form in which the coefficients are equal to constants (depending only on the absolute invariants) multiplied by powers of $(i_4 u + i_5 v)$. Conversely, any system of differential equations of form (I, D) with coefficients of this form satisfying the integrability conditions represents a family of projectively equivalent congruences with constant absolute invariants. W congruences of this type depend on an additional constant independent of the absolute invariants. Any congruence of the first type has all its Laplace transforms of the same type, but projectively distinct. Both sheets of the focal surface and the

cuspidal edges of all the developable surfaces have constant absolute invariants, except in the case of some of the W congruences.

In the second case (i. e., if $(\overset{(z)}{\mathfrak{B}} - \overset{(y)}{\mathfrak{B}}) = (\overset{(y)}{\mathfrak{C}''} - \overset{(z)}{\mathfrak{C}''}) = 0$) the differential equations can be reduced by a suitable transformation of the variables to a set with constant coefficients depending on the absolute invariants and on one additional arbitrary constant. Conversely, every congruence whose differential equations have constant coefficients satisfying the integrability conditions has constant absolute invariants, with $(\overset{(z)}{\mathfrak{B}} - \overset{(y)}{\mathfrak{B}}) = (\overset{(y)}{\mathfrak{C}''} - \overset{(z)}{\mathfrak{C}''}) = 0$. Furthermore, every congruence of this type is a W congruence and is projectively equivalent to its 1st and (-1)st Laplace transforms, corresponding lines being tangent to the common focal sheet at the same point; conversely, every congruence which is projective in this way to its 1st or (-1)st Laplace transform has constant absolute invariants, with $(\overset{(z)}{\mathfrak{B}} - \overset{(y)}{\mathfrak{B}}) = (\overset{(y)}{\mathfrak{C}''} - \overset{(z)}{\mathfrak{C}''}) = 0$. Every congruence of this type belongs to a quadratic complex. Both sheets of the focal surface and the cuspidal edges of all the developable surfaces have constant absolute invariants. The developable surfaces of each family are projective to one another.

A congruence of the third type, having $(\overset{(z)}{\mathfrak{B}} - \overset{(y)}{\mathfrak{B}}) = 0$ but $(\overset{(y)}{\mathfrak{C}''} - \overset{(z)}{\mathfrak{C}''}) \neq 0$, is determined (except for projective transformations) by one of its absolute invariants, j_1. The developable surfaces of each family are projective to one another. The 1st and (-1)st Laplace transforms are projectively distinct from one another and from the original congruence.

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ON THE PRIME DIVISORS OF THE CYCLOTOMIC FUNCTIONS[*]

BY

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Sylvester[†] gave the first theorem in which the prime divisors of the cyclotomic functions are distinguishable from the non-divisors by their linear character. T. Pepin[‡] in a later paper proved this statement of Sylvester, namely that all prime divisors of the function $x^3 - 3x \pm 1$, if integral values are assigned to x, are 3, or primes of the form $18n \pm 1$ exclusively. In a footnote to the above paper, Sylvester states the conjecture that the period function which gives rise to the equation for the determination of the e periods of order f of the primitive qth roots of unity, q a prime, is divisible by any power of a prime which is an eth power residue modulo q.

In the following paper we shall establish the above conjecture by Sylvester, giving in the form of a general theorem a test as to whether a given prime is a divisor or non-divisor of the general cyclotomic functions. In the development we shall need a theorem stated by Kummer[§] but not rigorously proved by him, as pointed out by H. J. S. Smith[||] and Dirichlet[¶], who both gave methods of correcting Kummer's error which are substantially the same as that given by Kummer himself in a later paper.[**] We shall give here an independent proof of the theorem to enable us to draw conclusion as to the ideal factors of the primes in the cyclotomic subfields.

Let q be an odd rational prime, and let ζ designate one of the primitive qth roots of unity. Let the domain of rationality defined by ζ be designated by $k(\zeta)$. Consider p a prime different from q and appertaining to the exponent f modulo q. Then f must be a divisor of $q-1$, so we write $q-1 = e \cdot f$. In $k(\zeta)$, p will be the product of e prime ideals each of degree f, hence we write $p = \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdots \mathfrak{p}_e$.

Let

(1) $$\eta_h = \zeta^{g^h} + \zeta^{g^{e+h}} + \zeta^{g^{2e+h}} + \cdots + \zeta^{g^{(f-1)e+h}}$$

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[†] Comptes Rendus, vol. 90 (1880), pp. 287-9.

[‡] Comptes Rendus, vol. 90 (1880), pp. 526-8.

[§] Journal für Mathematik, vol. 30 (1846), pp. 107-116.

[||] Report of British Association, 1860, p. 128, footnote.

[¶] Bulletin des Sciences Mathématiques, ser. 2, vol. 33, p. 54.

[**] Journal für Mathematik, vol. 53 (1857), p. 143.

be a Gaussian period of f generators; η_h will be a root of an equation $P(x) = 0$ with rational integral coefficients of degree e. The pth power of η_h will satisfy the congruence

$$(2) \qquad \eta_h^p \equiv \zeta^{pg^h} + \zeta^{pg^{e+h}} + \zeta^{pg^{2e+h}} + \cdots + \zeta^{pg^{(f-1)e+h}} \pmod{p}.$$

Now g is a primitive root of the congruence

$$(3) \qquad g^{q-1} \equiv 1 \pmod{q}$$

and the integers $g^1, g^2, g^3, \cdots, g^{q-1}$ form a reduced residue system of incongruent integers, modulo q, where we mean by such a system all the integers of a complete residue system, modulo q, which are prime to q. Every integer that is not divisible by q is congruent to one and only one of these powers of g, mod q, and since p is not equal to q we have $p \equiv g^t$, mod q, where t is an integer of the set $1, 2, 3, \cdots, q-1$. Furthermore t must be a multiple of e, since p appertains to the exponent f and g to the exponent $q-1$, mod q, and raising both sides of the last congruence to the power f we get on comparison the two resulting relations $g^{t\cdot f} \equiv g^{e\cdot f}$, mod q, which is possible when and only when $t\cdot f \equiv e\cdot f$, mod $q-1$. Whence, since $q-1 = e\cdot f$, we conclude $t \equiv 0$, mod e. Therefore we can write $p \equiv g^{s\cdot e}$, mod q. Multiplying both sides of this congruence by $g^{k\cdot e+h}$, we have

$$(4) \qquad p\cdot g^{k\cdot e+h} \equiv g^{(k+s)e+h} \pmod{q}.$$

Now $(k+s)e+h \equiv r\cdot e+h$, mod $q-1$, and since g is a primitive number,

$$(5) \qquad g^{(k+s)e+h} \equiv g^{r\cdot e+h} \pmod{q}.$$

Here $r < f$ and $r\cdot e+h$ will appear somewhere among the integers $0, 1, 2, \cdots, q-2$. Then combining (4) and (5) we have

$$(6) \qquad p\cdot g^{k\cdot e+h} \equiv g^{r\cdot e+h} \pmod{q}.$$

Let k run over the set of integers $0, 1, 2, \cdots, f-1$; then r will also run over the same set of integers, since r is less than f. As k varies over this set, r will vary over the same set in a different order and no two distinct values of k will give the same r; for suppose we could have, say,

$$p\cdot g^{k_1\cdot e+h} \equiv g^{r_1\cdot e+h} \pmod{q},$$

and

$$p\cdot g^{k_2\cdot e+h} \equiv g^{r_1\cdot e+h} \pmod{q}.$$

Then we must have

$$p \cdot g^{k_1 \cdot e+h} \equiv p \cdot g^{k_2 \cdot e+h} \pmod{q}.$$

We may divide out p, since p and q are by hypothesis relatively prime; hence we have $g^{k_1 \cdot e+h} \equiv g^{k_2 \cdot e+h}$, $\bmod q$. From this it follows that $k_1 \equiv k_2$, $\bmod f$, which since k_1 and k_2 are both less than f is possible only when $k_1 = k_2$. Then each power of the set $0 \cdot e + h$, $1 \cdot e + h$, $2 \cdot e + h$, $\cdots$, $(f-1)e+h$ will appear once and only once in the resulting system of exponents reduced $\bmod q-1$. Hence if we apply this reduction to each of the powers of the ζ's in (2) they will each go over into some one of the powers of the ζ's appearing in the period η_h and no two will be repeated. Hence we have exactly

$$\eta_h^p \equiv \zeta^{g^h} + \zeta^{g^{e+h}} + \zeta^{g^{2e+h}} + \cdots + \zeta^{g^{(f-1)e+h}} \pmod{p} \tag{7}$$

or

$$\eta_h^p \equiv \eta_h \pmod{p}. \tag{8}$$

Now $\mathfrak{p}_i$ is an ideal factor of p in $k(\zeta)$; hence in $k(\zeta)$ we have

$$\eta_h^p \equiv \eta_h \pmod{\mathfrak{p}_i}.$$

Also η_h is a generating number of $k(\eta)$; hence $\mathfrak{p}_i$ will be of the first degree in $k(\eta)$. Therefore p will be in $k(\eta)$ the product of e prime ideals each of the first degree, since the subscript i may run over the set of integers 1, 2, 3, $\cdots$, e. Hence we have the following

THEOREM 1. *If p is a rational prime different from the rational prime q and appertaining to the exponent f, modulo q, and $q-1 = e \cdot f$, then in $k(\eta)$, the domain generated by η_h, a root of the Gauss period equation of degree e, p is the product of e prime ideals each of the first degree.*

We now take up the application of the preceding results to investigate some of the properties of the prime divisors of the general cyclotomic period function, ascertaining a means of distinguishing the divisors from the non-divisors.

Consider, as before, q any odd rational prime and let $\eta_0, \eta_1, \cdots, \eta_{e-1}$ be the e periods of order $f = (q-1)/e$ of the primitive qth roots of unity. The domain $k(\zeta)$ is an abelian domain and hence the sub-domain $k(\eta)$ is also an abelian domain, since every sub-domain of a cyclotomic domain is a cyclotomic domain and every cyclotomic domain is an abelian domain. Let x take on an integral value "a" and suppose $P(a)$ to be factored into its rational prime factors as follows: $P(a) = p_1^{e_1} \cdot p_2^{e_2} \cdots p_k^{e_k}$. Suppose p_i

is any one of these rational prime divisors of $P(a)$; then we have $P(a) \equiv 0$, $\bmod p_i$. Hence $P(x) \equiv 0$, $\bmod p_i$, has a solution in $k(1)$, and we write

$$P(x) \equiv (x-a)\cdot Q(x) \pmod{p_i}. \tag{9}$$

E. Netto* has shown that the essential divisor of the discriminant of the field defined by one of the e periods of the primitive qth roots of unity, q a prime, is q^{e-1}. Now we consider p_i as different from q and also not an unessential discriminantal divisor; therefore p_i cannot contain a power of a prime ideal in $k(\eta)$ as a factor. Now from (9) we see that p_i must have a prime ideal factor $\mathfrak{p}$ of the first degree in $k(\eta)$, since p_i is not an unessential discriminantal divisor. Then for every integer α of the domain, $\alpha^{p_i} \equiv \alpha$, mod $\mathfrak{p}$. The domain $k(\eta)$ is an abelian domain; let G be the group of the domain. If we apply a substitution of G, $\mathfrak{p}$ will go over into $\mathfrak{p}'$, and α into α'. Hence we will have the relation $\alpha'^{p_i} \equiv \alpha'$, mod $\mathfrak{p}'$, since if $\alpha^{p_i-1}-1$ is a number of $\mathfrak{p}$, after the substitution is applied $\alpha'^{p_i-1}-1$ will be a number of $\mathfrak{p}'$. This will be true for every integer of the domain, since α represented any integer of the domain; hence $\mathfrak{p}'$ is a prime ideal factor of p_i of the first degree. Now we can apply each of the e substitutions of G, and since p_i cannot contain a power of a prime ideal the resulting ideal factors will all be different from each other and each of the first degree. Then p_i will be the product of e prime ideals all of the first degree in $k(\eta)$.

Now in passing to the higher domain $k(\zeta)$, p_i will be the product of e or more ideals each of degree not greater than f, since some of the prime ideal factors of p_i in $k(\eta)$ may break up into further factors when we pass to $k(\zeta)$ or they may maintain their prime character and increase their degree. Such degree will not exceed f, since the sum of the degrees of the factors will not exceed the degree of the field. The necessary and sufficient condition that p_i resolve into factors of degree f in $k(\zeta)$ is that p_i appertain to f, modulo q. But the degree of no one of the ideal factors of p_i in $k(\zeta)$ can exceed f, hence p_i cannot appertain to an exponent greater than f. If p_i appertain to an exponent less than f, we shall show that such exponent must be a factor of f and hence of $(q-1)/e$. Let $\bar{e}$ be the number of factors into which p is decomposed when we pass to $k(\zeta)$ and $\bar{f}$ the degree of each factor. Then we have $\bar{e}\cdot\bar{f} = q-1 = e\cdot f$. Now since $\bar{e}$ is the number of factors, if we suppose that each $\mathfrak{p}$ is split up into σ factors when passing to $k(\zeta)$ we have $\bar{e} = e\sigma$, whence $e\sigma\bar{f} = e\cdot f$, or $\sigma\bar{f} = f$. That is, $\bar{f}$ is a factor of f. Hence it follows in any case that $p_i^f \equiv 1$, mod q. We now have

* Mathematische Annalen, vol. 24 (1884), p. 579.

THEOREM II. *Let $P(x) = 0$ be the equation which has as its roots the e periods $\eta_0, \eta_1, \cdots, \eta_{e-1}$ of order f of the primitive qth roots of unity, where q is a prime, and let "a" be an integral value of x such that $P(a) = p_1^{e_1} \cdot p_2^{e_2} \cdots p_i^{e_i} \cdots p_k^{e_k}$, where $p_i (i = 1, 2, 3, \cdots, k)$ is a rational prime which is not a divisor of the discriminant of $P(x) = 0$; then p_i must satisfy the congruence* $p_i^{(q-1)/e} \equiv 1$, mod q.

Conversely, we have, from Theorem I, if p_i appertains to an exponent $(q-1)/e$, then p_i will be in $k(\eta)$ the product of e prime ideals each of the first degree, and therefore the congruence $P(x) \equiv 0$, mod p_i, has e solutions in $k(1)$ so that there must exist at least e values of "a" such that p_i will be found somewhere among the divisors of $P(x)$.

If p_i appertains to an exponent which is a factor of f, say to $\bar{f}$, then $\bar{e}$ will be a multiple of e. Form the period

$$\bar{\eta}_h = \zeta^{g^h} + \zeta^{g^{\bar{e}+h}} + \zeta^{g^{2\bar{e}+h}} + \cdots + \zeta^{g^{(\bar{f}-1)\bar{e}+h}}.$$

Let $\bar{e} = c \cdot e$; then c must be a factor of f. Then if we form the e periods, each one of these will be the sum of c $\bar{e}$ periods, hence the field $k(\eta)$ is a sub-field of $k(\bar{\eta})$. In the field $k(\bar{\eta})$ we have, from Theorem I, p_i the product of $\bar{e}$ prime ideals each of the first degree, so that, in passing to the sub-field $k(\eta)$, p_i will be the product of e prime ideals each of the first degree, because if the divisors of p_i are of the first degree in a field the divisors in a sub-field are necessarily of the first degree. In this case the congruence $P(x) \equiv 0$ will have e integral solutions and p_i will be found among the divisors of $P(x)$.

We may then classify all primes as to their character as divisors or non-divisors of the general cyclotomic function for the e periods of the primitive qth roots of unity. Those primes which belong to an exponent greater than $f = (q-1)/e$, except the primes that are divisors of the discriminant of the equation $P(x) = 0$, and all primes which belong to an exponent less than f but not a factor of f, will not be found as divisors of the function $P(x)$. But those primes which belong to an exponent $(q-1)/e$, or to an exponent which is a factor of $(q-1)/e$, will be found somewhere among the divisors of $P(x)$. We may state this result in the form of a general theorem.

THEOREM III. *A necessary and sufficient condition that p shall be a prime divisor of the cyclotomic function $P(x)$ is that it satisfy the congruence* $p^{(q-1)/e} \equiv 1$, mod q, *except for those primes which are divisors of the discriminant of $P(x) = 0$.*

It is evident from Theorem III that the conjecture of Sylvester is correct, since this is also a necessary and sufficient condition that p be an eth power residue modulo q.

There are certain forms which are associated with the period equations, and which are obtained from the period equations by linear transformation, which possess properties as the above with certain exceptions which are introduced by the transformation and which can be determined. These forms are important from the standpoint of their simplicity and applicability of the results found. Let p be a prime of the form $6 \cdot n + 1$. We can build three periods of the primitive pth roots of unity of order $(p-1)/3$ and the cubic equation having these periods as its roots is found to be*

$$x^3 + x^2 - \frac{p-1}{3} \cdot x - \frac{1}{9}\left(p \cdot a + \frac{p-1}{3}\right) = 0,$$

which by the transformation $y = 3x + 1$ takes the form

$$y^3 - 3py - pA = 0.$$

where $4 \cdot p = A^2 + 27 B^2$ and $A \equiv 1$, mod 3; $B \equiv 0$, mod 3. The prime divisors of this function satisfy the congruence $p_i^{(p-1)/3} \equiv +1$, mod p_i, with certain exceptions which are brought in from the transformation that was made upon the period. The discriminant of the transformed cubic is $27(4p^3 + p^2 A^2)$ which contains the unessential discriminantal divisor 3^3. The essential divisor of the discriminant is p^2, hence the prime divisors which may occur and yet not satisfy the relation as above given are the discriminantal divisors 3 and p.

If we consider primes of the form $4n + 1$, the quartic having the four periods of order $(p-1)/4$ as its roots is†

$$x^4 + x^3 - 3\left(\frac{p-1}{8}\right) \cdot x^2 - \frac{p(a+1) + \frac{p-1}{2}}{8} \cdot x - \frac{p(a+1)^2 - \left(\frac{p-1}{2}\right)^2}{64} = 0,$$

which by the transformation $y = 4x + 1$ becomes

$$(y^2 - p)^2 - 4p(y + a)^2 = 0.$$

Here we find the unessential discriminantal divisor 2 entering because of the transformation, hence with the exception of the divisors p and 2, all other primes which are not divisors of the discriminant of the field may be classed as divisors or non-divisors of the function $(y^2 - p)^2 - 4p(y + a)^2$ according as they satisfy the congruence $p_i^{(p-1)/4} \equiv 1$, mod p, or do not satisfy this relation.

* Gauss, *Disquisitiones Arithmeticae,* Art. 359.

† Bachmann, *Die Lehre von der Kreistheilung,* p. 228.

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ON THE ROOTS OF THE RIEMANN ZETA FUNCTION*

BY

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The object of the following paper is to simplify the methods and formulas developed and used by Gram,† Lindelöf,‡ and Backlund,§ in numerical investigations connected with the roots of the Zeta function. I apply these to locating and calculating additional roots.

I start with the formulas, as given by Backlund,

$$(1) \qquad \zeta(s) = \sum_{\nu=1}^{n-1} \nu^{-s} + \frac{1}{2} n^{-s} + \frac{n^{1-s}}{s-1} + \sum_{\nu=1}^{k} T_\nu + R_k,$$

$$(2) \qquad T_\nu = (-1)^{\nu-1} \frac{B_\nu}{(2\nu)!} \frac{s(s+1)\cdots(s+2\nu-2)}{n^{s+2\nu-1}}.$$

$$(3) \qquad |R_k| < \left|\frac{s+2k+1}{\sigma+2k+1}\right| |T_{k+1}|, \qquad s = \sigma + ti,$$

and, when $\sigma = \frac{1}{2}$,

$$(4) \qquad \begin{aligned} |T_\nu| &= \frac{\alpha_\nu}{\sqrt{n}} \left(\frac{t}{n}\right)^{2\nu-1}, \\ \alpha_\nu &= \frac{B_\nu}{(2\nu)!} \sqrt{\left(1+\frac{1}{4t^2}\right)\left(1+\frac{9}{4t^2}\right)\cdots\left(1+\frac{(4\nu-3)^2}{4t^2}\right)}. \end{aligned}$$

Formula (3) often gives too large an upper limit for $|R_k|$. A smaller limit can generally be obtained, as suggested by Lindelöf, by using

$$(5) \qquad |R_k| < |T_{k+1}| + |T_{k+2}| + \cdots + |T_l| + |R_l|,$$

with a suitable choice for l. To calculate an upper limit for a given remainder with any degree of precision by means of (3) and (5) is quite laborious. To shorten the work, determine the ratio of $|T_{\nu+1}|$ to $|T_\nu|$ by means of (4):

$$(6) \qquad \left|\frac{T_{\nu+1}}{T_\nu}\right| = \frac{B_{\nu+1}}{B_\nu}\left(\frac{t}{n}\right)^2 \frac{\sqrt{\left[1+\left(\frac{4\nu-1}{2t}\right)^2\right]\left[1+\left(\frac{4\nu+1}{2t}\right)^2\right]}}{(2\nu+1)(2\nu+2)}.$$

* Presented to the Society, October 25, 1924.

† *Note sur les zéros de la fonction $\zeta(s)$ de Riemann*, Acta Mathematica, vol. 27 (1903), p. 289.

‡ *Sur une formule sommatoire générale*, Acta Mathematica, vol. 27, p. 305.

§ R. J. Backlund, *Ueber die Nullstellen der Riemannschen Zetafunktion*, Dissertation, Helsingfors, 1916.

Consider the identity

$$\left[1+\left(\frac{4\nu-1}{2t}\right)^2\right]\left[1+\left(\frac{4\nu+1}{2t}\right)^2\right] = \left[1+\frac{16\nu^2+1}{4t^2}\right]^2-\left(\frac{2\nu}{t^2}\right)^2.$$

If t is large in comparison with ν, as will be the case in what follows, the last term is very small and may be omitted. The error thus introduced does not ordinarily affect the first seven decimal places. This gives in place of (6) the much simpler approximate formula

$$(7) \qquad |T_{\nu+1}| = b_\nu\left[1+\frac{(4\nu)^2+1}{4t^2}\right]\left(\frac{t}{n}\right)^2|T_\nu|,$$

in which

$$b_1 = \frac{1}{60}, \quad b_2 = \frac{1}{42}, \quad b_3 = \frac{1}{40}, \quad b_4 = \frac{10}{396},$$

$$\begin{aligned} b_5 &= .025\,311\,355, \\ b_6 &= .025\,325\,615, \\ b_7 &= .025\,329\,132, \\ b_8 &= .025\,330\,005\,5, \\ b_9 &= .025\,330\,223, \\ b_{10} &= .025\,330\,278, \\ b_{11} &= .025\,330\,291, \\ b_{12} &= .025\,330\,295, \\ b_{13} &= .025\,330\,296. \end{aligned}$$

These coefficients are evidently converging to a limit. In fact if we use in

$$b_\nu = \frac{B_{\nu+1}}{B_\nu}\cdot\frac{1}{(2\nu+1)(2\nu+2)}$$

the relations

$$B_\mu = \frac{(2\mu)!}{2^{2\mu-1}\pi^{2\mu}}\cdot\zeta(2\mu), \qquad \lim_{\mu\to\infty}\zeta(2\mu) = 1,$$

we obtain

$$\lim_{\nu\to\infty} b_\nu = \frac{1}{4\pi^2} = .025\,330\,295\,91.$$

To formula (7) should be joined the first formula (4), namely

$$(7') \qquad |T_1| = \frac{\sqrt{t^2+\frac{1}{4}}}{12\,n^{3/2}}.$$

It is desirable to have some simple method for determining the value for l in (5) that will give the lowest upper limit for $|R_k|$. Suppose this limit determined from two consecutive values of l, $l = \lambda$ and $l = \lambda + 1$, and assume that the right member of (5) is less for $l = \lambda + 1$ than it is for $l = \lambda$. This leads to the inequality

$$|R_{\lambda+1}| + |T_{\lambda+1}| < |R_\lambda|,$$

in which $|R_\nu|$ is used to indicate, not the actual numerical value of the remainder, but its upper limit. Accordingly, replace $|R_{\lambda+1}|$ and $|R_\lambda|$ by the right members of (3) and then replace $|T_{\lambda+2}|$ by (7). Put $b_{\lambda+1} = .02533$ and divide out the common factor $|T_{\lambda+1}|$. In the resulting formula drop the three fractional terms having denominators $4t^2$, these being small in comparison with the other terms. The effect is to strengthen somewhat the inequality and we obtain the relation

$$\frac{.02533\,t}{2\lambda + \frac{7}{2}}\left(\frac{t}{n}\right)^2 + 1 < \frac{t}{2\lambda + \frac{3}{2}}, \tag{8}$$

from which to determine the largest possible value of λ when t and n are given. The easiest way to find λ from (8) is by trial. If the value obtained makes both members very nearly equal, the next lower integer should be taken for λ, giving $l = \lambda + 1$ as the best value to use in (5).

Suppose $t' > t$ and $n' > n$ are two other numbers such that

$$\frac{t'}{n'} = \frac{t}{n}. \tag{9}$$

Denote by T' and R' the new values of T and R. Then from (4) and (9) we deduce

$$|T'_\nu| < \sqrt{\frac{n}{n'}}\,|T_\nu|, \tag{10}$$

and from (3) and (10) follows

$$|R'_k| < \sqrt{\frac{n'}{n}}\,|R_k|. \tag{11}$$

Combining (5), (10), and (11) we obtain, finally, the very useful formula

$$|R'_k| < \sqrt{\frac{n}{n'}}\,|R_k| + \left[\sqrt{\frac{n'}{n}} - \sqrt{\frac{n}{n'}}\right]|R_l|. \tag{12}$$

The upper limit determined for $|R'_k|$ by (12) is of course applicable for any t between t and t' if n remains fixed at the value n'.

If we write $\zeta(s)$ in the form

$$\zeta(s) = \varrho e^{i\varphi} = \varrho \cos\varphi + i\varrho \sin\varphi \tag{13}$$

and take $s = \frac{1}{2} + ti$, then ϱ and φ are functions of t, the latter of which was obtained by Gram in a very simple approximate form. Following the example of Gram, I will denote the real and imaginary components of $\zeta(\frac{1}{2} + ti)$ by $C(t)$ and $S(t)$ respectively, so that

$$C(t) = \varrho(t)\cos\varphi(t), \qquad S(t) = \varrho(t)\sin\varphi(t).$$

Further, the roots of $\cos\varphi(t) = 0$ will be represented by β_n, the roots of $\sin\varphi(t) = 0$ by γ_n, and the roots of $\varrho(t)$, which are the roots of the Zeta function, by α_ν. Gram calculated the first fifteen of the roots α and called attention to the fact that the α's and the γ's separate each other. I will refer to this property of the roots as *Gram's Law*. Gram expressed the belief that this law is not a general one. It is one of the objects of this paper to establish that fact. For this purpose I make use of a theorem proved by Gram which states that if $C(t)$ takes the same sign when $t = \gamma_\nu$ and $t = \gamma_{\nu+1}$, then at least one root α occurs between these two values of t.

Taking the real terms in (1) with $\sigma = \frac{1}{2}$, we have

$$C(t) = K + C_0 + C_1 + \cdots + C_k + r_k, \tag{14}$$

in which

$$K = 1 + \sum_{\nu=2}^{n-1} \frac{1}{\sqrt{\nu}}\cos(t\log\nu) + \frac{1}{2\sqrt{n}}\cos(t\log n), \tag{15}$$

$$C_0 = C_0(t) = -\frac{\sqrt{n}}{t^2+\frac{1}{4}}\left(ts + \frac{1}{2}c\right). \qquad \begin{aligned} s &= \sin(t\log n), \\ c &= \cos(t\log n), \end{aligned}$$

$$C_1 = \frac{ts + \frac{1}{2}c}{12\, n^{3/2}}, \tag{16}$$

$$C_2 = \frac{\left(ts + \frac{1}{2}c\right) + 4c}{720\, n^{3/2}}\left(\frac{t}{n}\right)^2 - \cdots,$$

$$C_3 = \frac{\left(ts + \frac{1}{2}c\right) + 12c}{30240\, n^{3/2}} \left(\frac{t}{n}\right)^4 - \cdots.$$

(16)

$$C_4 = \frac{\left(ts + \frac{1}{2}c\right) + 24c}{1209600\, n^{3/2}} \left(\frac{t}{n}\right)^6 - \cdots.$$

$$|r_k| \leqq |R_k|.$$

I use only the principal terms in C_2, C_3, and C_4 as the omitted terms do not affect the degree of accuracy required in this work.

To test Gram's Law, the values of γ_ν are calculated by the formula

$$\frac{t}{2\pi}\left(\log\frac{t}{2\pi} - 1\right) = n + \frac{1}{8} \tag{17}$$

and substituted in (14)*. The value of t that satisfies (17) will be denoted by γ_ν, $\nu = n + 3$.

As a result of the computations carried as far as $\gamma_{140} = 300.468$, it is found that $C(\gamma_\nu)$ *is positive in every case except two, viz.,* $\gamma_{129} = 282.455$, *and* $\gamma_{137} = 295.584$. As γ_{129} is the test case for Gram's Law, the value of $C(\gamma_{129})$ has been carefully verified, using $n = 100$ to obtain its value with great accuracy. Using C_4 as the last term and employing five decimals in the computations, the result is

$$C(\gamma_{129}) = -.027. \qquad |R_4| < .00005.$$

A question that naturally arises is this. The value of γ_{129} has been determined by an approximate formula (17). Is it possible that its exact value would make $C(t)$ positive? By calculation I find $S(282.455) = -.00015$ with $|R_4| < .00005$. It is obvious that the error in γ_{129} is very slight and that a correction in its value which would cause $S(t)$ to vanish could not change the sign of $C(t)$. For $t = \gamma_{137}$, we find $C(t) = -.017$.

To locate the roots α in these cases in which Gram's Law fails, it is necessary to get values of t which change the sign of $C(t)$. For $t = 282.6$ we find $C(t) = +.279$, which shows that $C(t)$ has a root between γ_{129} and 282.6. This is an α since the nearest root of $\cos\varphi = 0$ is $\beta_{126} = 283.28$. Since $C(t)$ has opposite signs at γ_{129} and γ_{130}, there must be an even number

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* I am indebted to Dr. Jesse Osborne for carrying out most of these calculations. A Monroe calculating machine and the *Smithsonian Mathematical Tables* by Becker and Van Orstrand with the trigonometric functions of angles expressed in radian measure were indispensable adjuncts.

of roots α between these limits, according to Gram's theorem. Hence there must be two such roots at least, since one α has been found.

In like manner, since $C(295.4) = +.175$, there is a root α between $t = 295.4$ and γ_{137}, since the nearest β is $\beta_{133} = 294.76$. Hence there are two roots α, at least, in the interval $(\gamma_{136}, \gamma_{137})$. This gives the number of roots in the interval $(0, 300.468)$ as 138, at least, counting one root for each interval $(\gamma_\nu, \gamma_{\nu+1})$, with the exceptions just noted.

We now proceed to show that there are no other roots of $\zeta(s)$ in the region $0 < t < 300.468$. For this purpose the method of Backlund (with same modifications) is used. Let the number of zeros of $\zeta(s)$ for which $0 < t < T$ be denoted by $N(T)$. Then*

$$(18)\quad N(T) = \frac{T}{2\pi}\left(\log\frac{T}{2\pi} - 1\right) + \frac{7}{8} + \frac{1}{\pi}\Delta_{abc}\arg\zeta(s) + \frac{1}{\pi}R(T),$$

in which $\Delta_{abc}\arg\zeta(s)$ denotes the increment that $\arg\zeta(s)$ takes when s describes the broken line $a\,b\,c$ in the s-plane, starting at the point $a = \frac{3}{2}$ and moving along the straight line $\sigma = \frac{3}{2}$ to the point $b = \frac{3}{2} + iT$, thence along the straight line $t = T$ to the point $c = \frac{1}{2} + iT$. Moreover,

$$R(T) < \frac{1}{48T} + \frac{0.2}{T^3}.$$

Backlund proves that $\Delta_{abc}\arg\zeta(s)$ is numerically less than $\pi/2$, for the case $T = 200$, by proving that $\Re\zeta(s) = \varrho\cos\varphi$ does not vanish anywhere on the line $a\,b\,c$. It follows that $\cos\varphi$ does not vanish at any point of this line and hence that φ does not pass through $\pm\pi/2$, assuming $\varphi = 0$ at a. The first step consists in proving that $\cos\varphi$ does not vanish on the line $a\,b$, T being entirely arbitrary. The second part of Backlund's proof, while very simple and ingenious, takes advantage of the fact that $\Re\zeta(\frac{1}{2} + 200i) = C(200)$ has an unusually large value, viz., 4.6. In the case I am dealing with, $T = 300.468$, we have $C(T) = 2.15$, which is too small for use in Backlund's method of proof. I accordingly proceed to modify the method so as to make it applicable to a much wider range of values of T.

On the line bc we have $s = \sigma + iT$.

$$(19)\qquad \frac{1}{2} \leq \sigma \leq \frac{3}{2}.$$

* Backlund, p. 22.

Write $\Re\zeta(s)$ in the form

$$(20)\qquad \Re\zeta(\sigma+iT) = K(\sigma)+L(\sigma),$$

$$(21)\qquad K(\sigma) = \sum_{\nu=1}^{n-1} \nu^{-\sigma}\cos(T\log\nu)+\frac{1}{2}n^{-\sigma}\cos(T\log n),$$

$$(22)\qquad L(\sigma) = \Re\left(\frac{n^{1-s}}{s-1}+\sum_{\nu=1}^{k} T_\nu + R_k\right).$$

Each term in (21) has the property of decreasing numerically as σ increases. Denote such a term, $\mu^{-\sigma}\cos(T\log\mu)$, by g_μ, if it is positive, and by h_μ, if it is negative. The signs of the individual terms do not change in the interval (19).

Consider, now, a sum of positive and negative terms, $G(\sigma)+H(\sigma)$,

$$G(\sigma) = g_{\mu_1}+g_{\mu_2}+\cdots+g_{\mu_i},$$
$$H(\sigma) = h_{\nu_1}+h_{\nu_2}+\cdots+h_{\nu_j},$$

in which the indices are subject to the inequalities

$$(23)\qquad \mu_1<\mu_2<\cdots<\mu_i<\nu_1<\nu_2<\cdots<\nu_j.$$

Suppose further that the inequality

$$(24)\qquad G(\sigma)+H(\sigma)>0$$

is satisfied when $\sigma=\frac{1}{2}$. Then *relation* (24) *holds throughout the interval* (19). For $G(\sigma)$ evidently satisfies the inequalities

$$\mu_i^{1/2-\sigma}G\left(\frac{1}{2}\right)<G(\sigma)<\mu_1^{1/2-\sigma}G\left(\frac{1}{2}\right),\qquad \sigma>\frac{1}{2}.$$

There accordingly exists a number α, depending on σ, such that

$$G(\sigma) = \alpha G\left(\frac{1}{2}\right),\qquad \mu_1<\alpha^{1/(1/2-\sigma)}<\mu_i.$$

Similarly a number β exists such that

$$H(\sigma) = \beta H\left(\frac{1}{2}\right),\qquad \nu_1<\beta^{1/(1/2-\sigma)}<\nu_j,$$

whence, from (23), and since $\frac{1}{2}-\sigma<0$, follows $\beta<\alpha$. Hence we obtain the relation

$$G(\sigma)+H(\sigma) = \alpha\left[G\left(\frac{1}{2}\right)+H\left(\frac{1}{2}\right)\right]+(\beta-\alpha)\,H\left(\frac{1}{2}\right)>0.$$

Accordingly, if we can group the terms of $K(\frac{1}{2})$ into one or more sets of the form $G(\frac{1}{2})+H(\frac{1}{2})$ having the above properties and including all of the negative terms $h_\nu(\frac{1}{2})$, together with a sufficient number of positive terms $g_\mu(\frac{1}{2})$ to insure that each set is positive, then $K(\sigma)>0$ in the interval (19). If there are any unused positive terms of $K(\frac{1}{2})$, we endeavor to group them with negative terms occurring in $L(\frac{1}{2})$ so that each group shall be positive throughout (19).

Apply now to the case $T=\gamma_{140} = 300.468$, $n = 51$. For the terms in $K(\frac{1}{2})$ I obtain the following results, in which the notation $(\mu_1, \mu_2, \cdots)$ means $g_{\mu_1}(\frac{1}{2})+g_{\mu_2}(\frac{1}{2})+\cdots$, while $-(\nu_1, \nu_2, \cdots)$ stands for $h_{\nu_1}(\frac{1}{2})+h_{\nu_2}(\frac{1}{2})+\cdots$;

$$(1, 2)-(3, 4, 6, 8) = +.235,$$
$$(5, 7, 9, 10)-(11, 13, 15, 16, 17, 20, 21) = +.130,$$
$$(12, 14, 18, 19, 22, 23, 24, 25, 26)$$
$$-(27, 28, 30, 32, 34, 36, 37, 40, 41, 42) = +.114.$$

All the negative terms of $K = K(\frac{1}{2})$ have now been used.

The first term of $L(\sigma)$ is negative when $\sigma=\frac{1}{2}$. We easily deduce the inequality

$$\left|\Re\left(\frac{n^{1-s}}{s-1}\right)\right| = \left|\frac{(\sigma-1)\,c-Ts}{[(\sigma-1)^2+T^2]\,n^{\sigma-1}}\right|, \qquad c = \cos(T\log n), \quad s = \sin(T\log n),$$

$$< \frac{A}{n^\sigma}, \qquad A = \frac{\frac{1}{2}|c|+T|s|}{T^2}\,n,$$

which holds for all values of σ in (19). If we can find a sum of unused terms $G(\sigma)$ of $K(\sigma)$ such that $G(\sigma)-An^{-\sigma}>0$ for $\sigma=\frac{1}{2}$, then this inequality will hold throughout the interval (19). In the present case we find $An^{-1/2}=.004$ while $g_{29}(\frac{1}{2}) = .183$, whence it follows that

$$g_{29}(\sigma)+\Re\left(\frac{n^{1-s}}{s-1}\right)>0$$

throughout (19).

For our present purposes it is unnecessary to discuss the remaining terms of (22), with the exception of the remainder which will be denoted by $R_k(\sigma)$. I find that a sufficient condition for the existence of the inequality

$$G(\sigma) + R_k(\sigma) > 0 \tag{25}$$

throughout (19) is

$$G\left(\frac{1}{2}\right) - Q_k \left| R_k\left(\frac{1}{2}\right) \right| > 0, \tag{26}$$

$$Q_k = \sqrt{\frac{T^2 + \left(2k + \frac{5}{2}\right)^2}{T^2 + \frac{1}{4}}}.$$

In the present case, taking $k = 0$, we find $|R_0(\frac{1}{2})| < .586$, while the sum of the remaining terms of $K(\frac{1}{2})$ is 1.492. As Q_k is obviously but slightly greater than 1, it is unnecessary to calculate its value to assure ourselves that (26) is abundantly satisfied and hence (25). We thus find that $\Re\zeta(s)$ is positive along the line bc. Hence $\Delta_{abc} \arg \zeta(s) < \pi/2$.

Returning to formula (18), we obtain the results

$$\frac{T}{2\pi}\left(\log\frac{T}{2\pi} - 1\right) = 137.125,$$

$$\frac{1}{\pi} R(T) < .00003,$$

$$N(T) = 138 \pm \varepsilon,$$

$$\varepsilon = |\Delta_{abc} \arg \zeta(s) + R(T)| < .50003.$$

Since $N(T)$ is an integer, the only solution is $N(T) = 138$. As we have already located 138 roots α on the line $\sigma = \frac{1}{2}$, there are no other roots of $\zeta(s)$ in the region $0 < t < 300.468$.

If we wish to determine the number of roots in a larger interval, how shall we choose T without too much labor so that the above scheme is workable? Observation shows that $T = \gamma_\nu$ is likely to be a suitable choice. If by trial of the first terms of $C(\gamma_\nu)$ the choice of T is found unsuitable, the next adjacent γ is more than likely to answer the purpose. In the 121 cases in which all the terms of $C(\gamma_\nu)$ have been computed,

there are 68 cases in which the proposed scheme is applicable. To see just how it works out in practise, I have tried it on the case $T = 500$. The nearest γ is $\gamma_{272} = 500.593$. We start out with the calculation of some of the initial terms of $C(t)$ and find them to be $1 + .114 - .568 - .474 + .065 + 007 + \cdots$. We already find that the excess of positive over negative terms has almost disappeared. Accordingly try $\gamma_{271} = 499.157$. The first terms start off so favorably that the calculation is continued to the end and it is readily found that all the terms including C_0 and R_0 can be arranged in positive groups in a way to insure that the function $\Re\zeta(\sigma + iT)$ will remain positive in the interval (19) and hence

$$\Delta_{abc} \arg \zeta(s) < \frac{\pi}{2}$$

for $T = \gamma_{271}$. From (18) we obtain the result: *The number of zeros of $\zeta(s)$ in the critical strip $0 < t < 500$, $0 \leqq \sigma \leqq 1$ is exactly* 269. This number satisfies the Riemann formula

$$N(T) = \frac{T}{2\pi}\left(\log \frac{T}{2\pi} - 1\right) + \frac{7}{8}.$$

In fact this is exactly what the formula for the number of roots would become if we suppose that it is always possible to find a T, however large, such that $\Delta_{abc} \arg \zeta(s) < \pi/2$, since $R(T)$ in (18) is very small.

Gram has remarked on the strong tendency of $C(t)$ to take positive values and ascribes this to the fact that the series starts out with a large positive term $+1$. He expressed the belief, however, that the equilibrium would eventually be restored, and that $C(\gamma_\nu)$ would not always be positive. How slow $C(\gamma_\nu)$ has been to take a negative value, we have already seen. There is another law, observed by Dr. Osborne, which gives a still larger surplus in favor of the positive terms. *The series $C(\gamma_\nu)$ always has a group G of consecutive positive terms beginning with ν (the index of summation) $= n_1$ and ending with $\nu = n_2$, these integers increasing with γ_ν in such a way that the ratios $\gamma_\nu : n_1$ and $\gamma_\nu : n_2$ are very nearly constant, the first lying between* 7 *and* 8, *and the second between* 5 *and* 5.7. Moreover, the sum of the terms in each group is practically constant, being situated between 1.256 and 1.38. (In all except 9 cases this sum is greater than 1.32.) The number of terms in G gradually increases from 6, when $\gamma = 73.635$, to 12 in $C(\gamma_{140})$, and 16 in $C(\gamma_{271})$.

The following new roots of the Zeta function have been calculated by use of the series

$$S(t) = -\sum_{\nu=2}^{n-1} \frac{1}{\sqrt{\nu}} \sin(t \log \nu) - \frac{1}{2\sqrt{n}} \sin(t \log n)$$

$$+ \frac{\sqrt{n}}{t^2 + \frac{1}{4}} \left(\frac{1}{2}s - tc\right) + \frac{tc - \frac{1}{2}s}{12 n^{3/2}} + \frac{\left(tc - \frac{1}{2}s\right) - 4s}{720 n^{3/2}} \left(\frac{t}{n}\right)^2 + \cdots$$

$$+ \frac{\left(tc - \frac{1}{2}s\right) - 12s}{30240 n^{3/2}} \left(\frac{t}{n}\right)^4 + \cdots + \frac{\left(tc - \frac{1}{2}s\right) - 24s}{1209600 n^{3/2}} \left(\frac{t}{n}\right)^6 + \cdots.$$

$$s = \sin(t \log n), \qquad c = \cos(t \log n).$$

Only the principal terms of those derived from T_2, T_3, T_4 are retained, the parts omitted being too slight in value to affect the results. All of the calculations have been made with five decimals. The third decimal in α_n has been estimated by linear interpolation and may not be exact in all cases. I have recalculated the values of α_{11} to α_{15}, given by Gram to only one decimal. The results are as follows:

$\alpha_{11} = 52.970$,	$\alpha_{21} = 79.337$,
$\alpha_{12} = 56.446$,	$\alpha_{22} = 82.910$,
$\alpha_{13} = 59.347$,	$\alpha_{23} = 84.734$,
$\alpha_{14} = 60.833$,	$\alpha_{24} = 87.426$,
$\alpha_{15} = 65.113$,	$\alpha_{25} = 88.809$,
$\alpha_{16} = 67.080$,	$\alpha_{26} = 92.494$,
$\alpha_{17} = 69.546$,	$\alpha_{27} = 94.651$,
$\alpha_{18} = 72.067$,	$\alpha_{28} = 95.871$,
$\alpha_{19} = 75.705$,	$\alpha_{29} = 98.831$.
$\alpha_{20} = 77.145$,	

In finding a first approximation to a required α the following observed law has been very useful. If α lies on the segment from γ_ν to $\gamma_{\nu+1}$, it divides this into two segments $\gamma_\nu \alpha$ and $\alpha \gamma_{\nu+1}$ such that the ratio of the first to the second is > 1 (or < 1) according as the ratio $C(\gamma_\nu) : C(\gamma_{\nu+1})$ is > 1 (or < 1). Moreover, the first ratio is large or small according as the second ratio is large or small.

The following table gives the values of $C(\gamma_\nu)$ in the interval $200 < t < 300$. This table, in conjunction with that published by Backlund, locates all roots α in the interval $0 < t < 300$. The values of $C(\gamma_\nu)$ were calculated solely for the purpose of determining their signs and hence their values

may not be very exact. A plus sign is used to indicate a large positive value in those cases in which it was unnecessary to complete the computation.

There is one root α situated between two consecutive values of γ_ν given in the table, with exception of the four cases discussed in the text in which one α lies in each of the intervals (282.455, 282.6), (282.6, 284.1), (294.0, 295.4), (295.4, 295.6).

ν	γ_ν	$C(\gamma_\nu)$	ν	γ_ν	$C(\gamma_\nu)$
82	201.5	0.6	112	254.0	3.7
83	203.3	1.2	113	255.7	0.9
84	205.1	0.6	114	257.4	2.6
85	206.9	3.6	115	259.1	0.8
86	208.7	2.0	116	260.8	0.1+
87	210.5	2.5	117	262.5	+
88	212.3	1.2	118	264.1	2.4
89	214.0	0.5	119	265.8	0.6
90	215.8	1.4	120	267.5	0.6
91	217.6	5.8	121	269.2	2.7
92	219.4	1.2	122	270.8	2.1
93	221.1	0.3	123	272.5	+
94	222.9	2.9	124	274.2	+
95	224.6	0.7	125	275.8	0.5
96	226.4	3.9	126	277.5	1.5
97	228.2	2.6	127	279.1	0.2
98	229.9	1.5	128	280.8	+
99	231.6	0.3	129	282.5	−0.027
100	233.4	0.9	130	284.1	1.4
101	235.1	5.4	131	285.8	1.9
102	236.9	0.8	132	287.4	1.0
103	238.6	1.5	133	289.0	1.9
104	240.3	1.3	134	290.7	4.3
105	242.0	1.7	135	292.3	1.6
106	243.8	0.9	136	294.0	0.8
107	245.5	+	137	295.6	−0.017
108	247.2	0.05+	138	297.2	3.4
109	248.9	0.8	139	298.8	3.2
110	250.6	0.8	140	300.5	2.2
111	252.3	3.2			

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A GENERALISATION OF THE RIEMANNIAN LINE-ELEMENT*

BY

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1. In a manifold of N dimensions and coördinate system x^i, let $P(x^i)$ and $Q(x^i+dx^i)$ be two points with infinitesimal coördinate differences. Our fundamental postulate is as follows:

POSTULATE. *The points P and Q define an invariant infinitesimal line-element ds, expressible as a function of $x^1, x^2, \cdots, x^N, dx^1, dx^2, \cdots, dx^N$.*

Obviously ds must be homogeneous of the first degree in the differentials, and we write

$$ds^2 = F(x^1, \cdots, x^N; dx^1, \cdots, dx^N), \tag{1.1}$$

where F is homogeneous of the second degree in the differentials.† We shall in general write

$$F(x^1, \cdots, x^N; \xi^1, \cdots, \xi^N) = F(x; \xi). \tag{1.2}$$

The further essential postulate in the differential geometry of Riemann is

$$F(x; dx) = g_{ij}\, dx^i\, dx^j, \tag{1.3}$$

where g_{ij} are functions of the coördinates only. In the present paper I wish to develope the more obvious deductions from (1.1), without assuming (1.3).

2. For a coördinate transformation $x^i = x^i(x'^1, \cdots, x'^N)$, we have, writing $\dot{x}^i = dx^i/dt$,

$$\dot{x}^i = \frac{\partial x^i}{\partial x'^j}\dot{x}'^j, \tag{2.1}$$

and therefore

$$\frac{\partial \dot{x}^i}{\partial \dot{x}'^j} = \frac{\partial x^i}{\partial x'^j}; \tag{2.2}$$

also

$$\frac{d}{dt}\frac{\partial x^i}{\partial x'^j} = \frac{\partial^2 x^i}{\partial x'^k\, \partial x'^j}\dot{x}'^k = \frac{\partial \dot{x}^i}{\partial x'^j}. \tag{2.3}$$

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† Cf. P. Finsler, *Über Kurven und Flächen in allgemeinen Räumen*, Dissertation, Göttingen, 1918, p. 33, and J. A. Schouten, *Der Ricci-Kalkül*, Berlin, 1924, p. 36.

If ψ be any invariant function of the coördinates and their first derivatives with respect to t,

$$\frac{\partial \psi}{\partial \dot{x}'^i} = \frac{\partial \psi}{\partial \dot{x}^j}\frac{\partial \dot{x}^j}{\partial \dot{x}'^i} = \frac{\partial \psi}{\partial \dot{x}^j}\frac{\partial x^j}{\partial x'^i}, \qquad \text{by (2.2).}$$

Thus $\partial \psi / \partial \dot{x}^i$ is a covariant vector. Also

$$\frac{\partial^2 \psi}{\partial \dot{x}'^i \partial \dot{x}'^j} = \frac{\partial}{\partial \dot{x}'^i}\left(\frac{\partial \psi}{\partial \dot{x}^k}\frac{\partial x^k}{\partial x'^j}\right) = \frac{\partial^2 \psi}{\partial \dot{x}^l \partial \dot{x}^k}\frac{\partial x^l}{\partial x'^i}\frac{\partial x^k}{\partial x'^j}.$$

Thus $\partial^2 \psi / \partial \dot{x}^i \partial \dot{x}^j$ is a covariant tensor of the second rank. Similarly $\partial^3 \psi / \partial \dot{x}^i \partial \dot{x}^j \partial \dot{x}^k$ is a covariant tensor of the third rank.

We shall call

$$f_{ij} = \frac{1}{2}\frac{\partial^2}{\partial \dot{x}^i \partial \dot{x}^j} F(x; \dot{x})$$

the *fundamental tensor*, noting that if (1.3) is true, $f_{ij} = g_{ij}$. Since F is homogeneous of the second degree in the derivatives of the coördinates, f_{ij} is homogeneous of zero degree. Therefore Euler's theorem gives

$$\frac{\partial f_{ij}}{d \dot{x}^k}\dot{x}^k = 0. \tag{2.4}$$

Also, obviously,

$$\frac{\partial f_{ij}}{\partial \dot{x}^k}\dot{x}^j = 0. \tag{2.5}$$

Using the homogeneity conditions we find

$$ds^2 = F(x; \dot{x})\, dt^2 = f_{ij}\, dx^i\, dx^j, \tag{2.6}$$

a formula analogous to (1.3).

Defining f^{ij} as the minor of the determinant $f = \| f_{mn} \|$ corresponding to f_{ij}, preceded by the proper sign and divided by f, we have

$$f^{ij} f_{kj} = \delta^i_k (= 1 \text{ for } i = k; = 0 \text{ for } i \neq k) \tag{2.7}$$

and the ordinary mode of proof establishes that f^{ij} is a contravariant tensor of the second rank. (Cf. Eddington, *Report on the Relativity Theory of Gravitation*, 1920, p. 35.)

3. Defining the geodesics as curves of stationary length, we obtain from the calculus of variations the equations

$$(3.1) \qquad G_i \equiv \frac{d}{dt}\frac{\partial\sqrt{F}}{\partial \dot{x}^i} - \frac{\partial\sqrt{F}}{\partial x^i} = 0,$$

where $F = F(x; \dot{x})$, which equations retain their form for transformation of the parameter. For any other system of coördinates x'^i,

$$G'_i = \frac{d}{dt}\frac{\partial\sqrt{F}}{\partial \dot{x}'^i} - \frac{\partial\sqrt{F}}{\partial x'^i},$$

or, since $\partial\sqrt{F}/\partial\dot{x}^i$ is covariant,

$$\begin{aligned} G'_i &= \frac{d}{dt}\left(\frac{\partial\sqrt{F}}{\partial\dot{x}^j}\frac{\partial x^j}{\partial x'^i}\right) - \frac{\partial\sqrt{F}}{\partial\dot{x}^j}\frac{\partial\dot{x}^j}{\partial x'^i} - \frac{\partial\sqrt{F}}{\partial x^j}\frac{\partial x^j}{\partial x'^i} \\ &= G_j\frac{\partial x^j}{\partial x'^i} + \frac{\partial\sqrt{F}}{\partial\dot{x}^j}\left(\frac{d}{dt}\frac{\partial x^j}{\partial x'^i} - \frac{\partial\dot{x}^j}{\partial x'^i}\right) \end{aligned}$$

and thus, by (2.3), G_i is a covariant vector.

An explicit form of the geodesic equations is obtained as follows:

For any curve, choose $t = s$, so that $F = 1$ along the curve. Then

$$\frac{1}{2}\frac{d}{ds}\frac{\partial F}{\partial\dot{x}^i} - \frac{1}{2}\frac{\partial F}{\partial x^i} = \frac{d\sqrt{F}}{ds}\frac{\partial\sqrt{F}}{\partial\dot{x}^i} + \sqrt{F}\left(\frac{d}{ds}\frac{\partial\sqrt{F}}{\partial\dot{x}^i} - \frac{\partial\sqrt{F}}{\partial x^i}\right)$$

and therefore

$$(3.2) \qquad G_i = \frac{1}{2}\frac{d}{ds}\frac{\partial F}{\partial\dot{x}^i} - \frac{1}{2}\frac{\partial F}{\partial x^i}.$$

But $F = f_{ij}\dot{x}^i\dot{x}^j$, and thus

$$\begin{aligned} G_i &= \frac{d}{ds}\left(f_{ij}\dot{x}^j + \frac{1}{2}\frac{\partial f_{jk}}{\partial\dot{x}^i}\dot{x}^j\dot{x}^k\right) - \frac{1}{2}\frac{\partial f_{jk}}{\partial x^i}\dot{x}^j\dot{x}^k \\ &= \frac{d}{ds}(f_{ij}\dot{x}^j) - \frac{1}{2}\frac{\partial f_{jk}}{\partial x^i}\dot{x}^j\dot{x}^k, \text{ by (2.5)}, \end{aligned}$$

and, again using (2.5), we obtain

$$(3.3) \qquad G_i = f_{ij}\ddot{x}^j + \begin{bmatrix} j\ k \\ i \end{bmatrix}\dot{x}^j\dot{x}^k,$$

where

$$2\begin{bmatrix} j\ k \\ i \end{bmatrix} = \frac{\partial f_{ij}}{\partial x^k} + \frac{\partial f_{ik}}{\partial x^j} - \frac{\partial f_{jk}}{\partial x^i}.$$

Hence

$$G^i = f^{il} G_l = \ddot{x}^i + \begin{Bmatrix} j\ k \\ i \end{Bmatrix} \dot{x}^j \dot{x}^k \tag{3.4}$$

where

$$\begin{Bmatrix} j\ k \\ i \end{Bmatrix} = f^{il} \begin{bmatrix} j\ k \\ l \end{bmatrix},$$

and if the curve is a geodesic, its equations take the well known form for parameter s,

$$\ddot{x}^i + \begin{Bmatrix} j\ k \\ i \end{Bmatrix} \dot{x}^j \dot{x}^k = 0. \tag{3.5}$$

The Christoffel symbol is homogeneous of zero degree in the first derivatives of the coördinates with respect to s.

4. The equations of Levi-Civita for parallel propagation of a vector along a curve may easily be modified to meet the case of our more general metric. Let

$$2\overline{\begin{bmatrix} j\ k \\ i \end{bmatrix}} = \frac{\partial f_{ij}}{\partial \dot{x}^k} + \frac{\partial f_{ik}}{\partial \dot{x}^j} - \frac{\partial f_{jk}}{\partial \dot{x}^i} = \frac{1}{2} \frac{\partial^3 F}{\partial \dot{x}^i \partial \dot{x}^j \partial \dot{x}^k},$$

$$\overline{\begin{Bmatrix} j\ k \\ i \end{Bmatrix}} = f^{il} \overline{\begin{bmatrix} j\ k \\ l \end{bmatrix}}.$$

Let Y^i be defined as

$$Y^i = \dot{X}^i + \overline{\begin{Bmatrix} j\ k \\ i \end{Bmatrix}} X^j \ddot{x}^k + \begin{Bmatrix} j\ k \\ i \end{Bmatrix} X^j \dot{x}^k, \tag{4.1}$$

where X^i is a contravariant vector given as a function of t along a curve $x^i = x^i(t)$. For the coördinate system x'^i,

$$\begin{aligned} Y'^i &= \dot{X}'^i + \frac{1}{2} f'^{il} X'^j \left\{ \frac{\partial f'_{lj}}{\partial \dot{x}'^k} \ddot{x}'^k + \left(\frac{\partial f'_{lj}}{\partial x'^k} + \frac{\partial f'_{lk}}{\partial x'^j} - \frac{\partial f'_{jk}}{\partial x'^l} \right) \dot{x}'^k \right\} \\ &= \dot{X}'^i + \frac{1}{2} f'^{il} X'^j \left\{ \frac{d}{dt} (f'_{lj}) + \left(\frac{\partial f'_{lk}}{\partial x'^j} - \frac{\partial f'_{jk}}{\partial x'^l} \right) \dot{x}'^k \right\} \\ &= \frac{d}{dt} \left(X^m \frac{\partial x'^i}{\partial x^m} \right) + \frac{1}{2} f'^{il} X'^j \left\{ \frac{d}{dt} \left(f_{mn} \frac{\partial x^m}{\partial x'^j} \frac{\partial x^n}{\partial x'^l} \right) \right. \\ &\qquad \left. + \left[\frac{\partial}{\partial x'^j} \left(f_{mn} \frac{\partial x^m}{\partial x'^l} \frac{\partial x^n}{\partial x'^k} \right) - \frac{\partial}{\partial x'^l} \left(f_{mn} \frac{\partial x^m}{\partial x'^j} \frac{\partial x^n}{\partial x'^k} \right) \right] \dot{x}'^k \right\}. \end{aligned}$$

Thus

$$
\begin{aligned}
Y'^i - Y^m \frac{\partial x'^i}{\partial x^m} = X^m \frac{d}{dt}\left(\frac{\partial x'^i}{\partial x^m}\right) + \frac{1}{2} f'^{il} X'^j \Big\{ & f_{mn} \frac{d}{dt}\left(\frac{\partial x^m}{\partial x'^j} \frac{\partial x^n}{\partial x'^l}\right) \\
& + f_{mn}\left[\frac{\partial}{\partial x'^j}\left(\frac{\partial x^m}{\partial x'^l} \frac{\partial x^n}{\partial x'^k}\right) - \frac{\partial}{\partial x'^l}\left(\frac{\partial x^m}{\partial x'^j} \frac{\partial x^n}{\partial x'^k}\right)\right] \dot{x}'^k \\
& + \frac{\partial f_{mn}}{\partial \dot{x}^p}\left[\frac{\partial \dot{x}^p}{\partial x'^j} \frac{\partial x^m}{\partial x'^l} \frac{\partial x^n}{\partial x'^k} - \frac{\partial \dot{x}^p}{\partial x'^l} \frac{\partial x^m}{\partial x'^j} \frac{\partial x^n}{\partial x'^k}\right] \dot{x}'^k \Big\}.
\end{aligned}
$$

But, by (2.5),

$$
\frac{\partial f_{mn}}{\partial \dot{x}^p} \frac{\partial x^n}{\partial x'^k} \dot{x}'^k = \frac{\partial f_{mn}}{\partial \dot{x}^p} \dot{x}^n = 0;
$$

hence, using (2.3),

$$
\begin{aligned}
Y'^i - Y^m \frac{\partial x'^i}{\partial x^m} &= X^m \frac{\partial \dot{x}'^i}{\partial x^m} + f'^{il} X'^j f_{mn} \frac{\partial \dot{x}^m}{\partial x'^j} \frac{\partial x^n}{\partial x'^l} \\
&= X^m \frac{\partial \dot{x}'^i}{\partial x^m} + f'^{il} X^p \frac{\partial x'^j}{\partial x^p} f'_{qr} \frac{\partial x'^q}{\partial x^m} \frac{\partial x'^r}{\partial x^n} \frac{\partial \dot{x}^m}{\partial x'^j} \frac{\partial x^n}{\partial x'^l} \\
&= X^p \left(\frac{\partial \dot{x}'^i}{\partial x^p} + \frac{\partial x'^j}{\partial x^p} \frac{\partial x'^i}{\partial x^m} \frac{\partial \dot{x}^m}{\partial x'^j}\right) \\
&= X^p \left(\frac{\partial \dot{x}'^i}{\partial x^p} - \frac{\partial x'^i}{\partial x^m} \frac{\partial \dot{x}^m}{\partial \dot{x}'^j} \frac{\partial \dot{x}'^j}{\partial x^p}\right) \\
&= 0, \text{ by (2.2)}.
\end{aligned}
$$

Thus Y^i is a contravariant vector, and we shall define parallel propagation of X^i by the equations

$$
(4.2) \qquad Y^i \equiv \dot{X}^i + \left|\begin{matrix} j & k \\ & i \end{matrix}\right| X^j \ddot{x}^k + \left\{\begin{matrix} j & k \\ & i \end{matrix}\right\} X^j \dot{x}^k = 0,
$$

which reduce to the equations of Levi-Civita when (1.3) is true.

5. We shall now proceed to the definition of *angle*. Let A be any point and p, q two curves emanating from A. Let P, Q be points on p, q respectively, such that the arcs AP, AQ are each equal to s. Let $PQ = \sigma$. We shall define the angle θ between p and q by the equation

$$
(5.1) \qquad \cos\theta = 1 - \frac{1}{2} \lim_{s \to 0} \frac{\sigma^2}{s^2}.
$$

If the coördinates of A are x^i and those of P, Q are $x^i + dx^i$, $x^i + \delta x^i$ respectively, we find

$$
(5.2) \qquad \cos\theta = 1 - \frac{1}{2} F\left(x;\ \frac{dx}{\sqrt{F(x; dx)}} - \frac{\delta x}{\sqrt{F(x; \delta x)}}\right).
$$

But the expression on the right retains the same value if $dx^1, \cdots, dx^N$ or $\delta x^1, \cdots, \delta x^N$ are replaced by quantities proportional to them, and therefore (5.2) defines the angle between the curves if $dx, \delta x$ are any infinitesimal displacements in the directions of the curves. Since F is homogeneous of the second degree in its directional arguments, the angle thus defined does not depend on the order in which the curves are considered.

There is, however, another definition of angle which extends the fundamental property of parallel propagation in Riemannian space into the more general type under consideration. If we are given two vectors X^i, Y^i at a point on a curve C, we define *the angle* $\Phi(X, Y; C)$ *between the vectors, with respect to* C, by

$$\cos \Phi(X, Y; C) = \frac{f_{ij} X^i Y^j}{\sqrt{f_{mn} X^m X^n \cdot f_{pq} Y^p Y^q}}, \tag{5.3}$$

where the directional arguments of the f's are the coördinate derivatives $\dot{x}^i$ along C. Now if X and Y undergo parallel propagation along C,

$$\begin{aligned} \frac{d}{dt}(f_{ij} X^i Y^j) &= -f_{ij}\left(\overline{\left\{\begin{matrix} l\ k \\ i \end{matrix}\right\}}\ddot{x}^k + \left\{\begin{matrix} l\ k \\ i \end{matrix}\right\}\dot{x}^k\right)(X^l Y^j + X^j Y^l) \\ &\qquad + \left(\frac{\partial f_{ij}}{\partial \dot{x}^k}\ddot{x}^k + \frac{\partial f_{ij}}{\partial x^k}\dot{x}^k\right) X^i Y^j \\ &= X^i Y^j \left[\left(\frac{\partial f_{ij}}{\partial \dot{x}^k} - \overline{\left[\begin{matrix} i\ k \\ j \end{matrix}\right]} - \overline{\left[\begin{matrix} j\ k \\ i \end{matrix}\right]}\right)\ddot{x}^k + \left(\frac{\partial f_{ij}}{\partial x^k} - \left[\begin{matrix} i\ k \\ j \end{matrix}\right] - \left[\begin{matrix} j\ k \\ i \end{matrix}\right]\right)\dot{x}^k\right] \\ &= 0. \end{aligned}$$

Similarly the denominator in (5.3) also has a zero derivative, and thus *the angle between two vectors, with respect to a curve, remains constant when both vectors undergo parallel propagation along the curve.*

The foregoing definition of angle with respect to a curve gives us at once a definition of *perpendicularity of* Y *with respect to* X, expressed by the relation

$$f_{ij} X^i Y^j = \frac{1}{2}\frac{\partial F(x; X)}{\partial X^j} Y^j = 0, \tag{5.4}$$

where the directional arguments of f_{ij} are the components of X. We say, then, that *two vectors are perpendicular with respect to one another* if

$$\frac{\partial F(x; X)}{\partial X^i} Y^i = \frac{\partial F(x; Y)}{\partial Y^i} X^i = 0. \tag{5.5}$$

This last idea leads to consideration of a type of principal direction in a two-dimensional space, non-existent for the Riemannian metric; *those directions may be termed principal which are perpendicular with respect to one another.*

As a simple illustration, let

$$ds = \sqrt[4]{dx^{1^4} + dx^{2^4}}\,. \tag{5.6}$$

Then the conditions (5.5) give

$$X^{1^3} Y^1 + X^{2^3} Y^2 = Y^{1^3} X^1 + Y^{2^3} X^2 = 0.$$

Therefore

$$Y^1 = \pm Y^2,$$
$$X^1 = \mp X^2,$$

and the differential equations of the principal directions are

$$dx^1 \pm dx^2 = 0. \tag{5.7}$$

As in Riemannian geometry, every null-direction is perpendicular to itself in the sense of (5.4).

6. In the case where $ds^2 = F(x;\, dx)$ is a function of the differentials only, as in (5.6), f_{ij} are independent of the coördinates x^i, and $\left\{ {jk \atop i} \right\} = 0$. Thus, by (3.5), the equations of the geodesics are

$$\frac{d^2 x^i}{ds^2} = 0, \tag{6.1}$$

whence

$$x^i = \lambda^i s + \alpha^i. \tag{6.2}$$

Therefore for such types of line-element, the axioms of connection and order hold, as well as those axioms of congruence which do not deal with angles. Planes exist and the euclidean axiom of parallels is true.

For parallel propagation along any geodesic, we find from (4.2) and (6.1)

$$X^i = \text{constant}. \tag{6.3}$$

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ELEMENTARY FUNCTIONS AND THEIR INVERSES*

BY

J. F. RITT

The chief item of this paper is the determination of all elementary functions whose inverses are elementary. The elementary functions are understood here to be those which are obtained in a finite number of steps by performing algebraic operations and taking exponentials and logarithms. For instance, the function

$$\tan [e^{z^2} - \log_z(1+\sqrt{z})] + [z^2 + \log \operatorname{arc} \sin z]^{1/2}$$

is elementary.

We prove that *if $F(z)$ and its inverse are both elementary, there exist n functions*

$$\varphi_1(z), \quad \varphi_2(z), \quad \cdots, \quad \varphi_n(z),$$

where each $\varphi(z)$ with an odd index is algebraic, and each $\varphi(z)$ with an even index is either e^z or $\log z$, *such that*

$$F(z) = \varphi_n \varphi_{n-1} \cdots \varphi_2 \varphi_1(z)$$

each $\varphi_i(z)$ $(i < n)$ being substituted for z in $\varphi_{i+1}(z)$. That every $F(z)$ of this type has an elementary inverse is obvious.

It remains to develop a method for recognizing whether a given elementary function can be reduced to the above form for $F(z)$. How to test fairly simple functions will be evident from the details of our proofs. For the immediate present, we let the general question stand.

The present paper is an addition to Liouville's work of almost a century ago on the classification of the elementary functions, on the possibility of effecting integrations in finite terms, and on the impossibility of solving certain differential equations, and certain transcendental equations, in finite terms.† Free use is made here of the ingenious methods of Liouville.

* Presented to the Society, October 25, 1924.

† Journal de l'École Polytechnique, vol. 14 (1833), p. 36; Journal für die reine und angewandte Mathematik, vol. 13 (1833), p. 93; Journal de Mathématiques, vol. 2 (1837), p. 56, vol. 3 (1838), p. 523, vol. 6 (1841), p. 1. For extensions of Liouville's work on differential equations, see Lorenz, Hansen, Steen and Petersen, Tidskrift for Mathematik, 1874–1876; Koenigsberger, Mathematische Annalen, 1886; Mordukhai-Boltovski, University of Warsaw Bulletin, 1909, 1910.

Reference should also be made to the classifications by Painlevé* and by Drach† of the solutions of algebraic differential equations.

In the course of our work we prove a set of lemmas of which some are not uninteresting in themselves. That of § 14 promises to be useful in settling other questions on the elementary functions. The result on functions with elementary inverses is a corollary of a very general theorem stated in § 23.

We precede the solution of our problem by a discussion which is designed to lend rigor to our work. This discussion is more explicit, on certain points of special importance in the present paper, than that in our paper *On the integrals of elementary functions*.‡ The formal parts of our work can probably be followed without a careful reading of these preliminaries.

I. ELEMENTARY FUNCTIONS. THEIR DIFFERENTIATION. LIOUVILLE'S PRINCIPLE

1. An analytic function of z will be said to be *analytic almost everywhere* if, given any element of the function $P(z-z_0)$,§ any curve

$$z = \varphi(\lambda) \qquad (0 \leq \lambda \leq 1),$$

where $\varphi(0) = z_0$, and any positive ε, there exists a curve

$$(1) \qquad z = \varphi_1(\lambda) \qquad (0 \leq \lambda \leq 1),$$

where $\varphi_1(0) = z_0$, such that

$$|\varphi_1(\lambda) - \varphi(\lambda)| < \varepsilon$$

for $0 \leq \lambda \leq 1$, and such that the element $P(z-z_0)$ can be continued along the entire curve (1). Roughly speaking, an element of the function, if it cannot be continued along a given path, can be continued along some path in any neighborhood of the given one.

2. An algebraic function u, given by an irreducible equation

$$(2) \qquad \alpha_0 u^m + \alpha_1 u^{m-1} + \cdots + \alpha_m = 0,$$

* *Leçons sur les Équations Différentielles, professées à Stockholm*, Paris, 1897, p. 487.

† Annales de l'École Normale Supérieure, vol. 34 (1898), p. 243.

‡ These Transactions, vol. 25 (1923), p. 211.

§ It is to be recalled that an analytic element $P(z-z_0)$ is a convergent series of positive powers of $z-z_0$.

where each α is a polynomial in z with constant coefficients, is analytic almost everywhere, because its singularities are isolated.

In what follows the algebraic functions will frequently be called *functions of order zero*, and the variable z a *monomial of order zero*.

3. The functions e^v and $\log v$, where v is any non-constant algebraic function, are called by Liouville *monomials of the first order*. It is seen directly that e^v is analytic almost everywhere. If v is analytic, and nowhere zero, along a given curve, $\log v$ is analytic along the curve. If v should vanish for some points (necessarily isolated) of the curve, there is a curve arbitrarily close to the given one on which v is everywhere different from zero. Thus $\log v$ is analytic almost everywhere.

More generally we shall say, following Liouville, that u is a *function of the first order* if it is not algebraic and if it satisfies an equation like (2) in which each α is a rational integral combination of monomials of orders zero and one, not all α's being zero.

We mean by this that, for some point z_0, the function u and each of the monomials in the α's have analytic elements which, when combined by multiplication and addition to form the first member of (2), yield an element with coefficients all zero. We may of course assume that α_0 is not identically zero.

4. Let Γ be any area in the complex plane, and suppose that we can continue the above mentioned element of u with center at z_0 into and all over Γ, so that u has a branch which is uniform and analytic throughout Γ. Let C be some curve along which u can be continued from z_0 into Γ. Any curve which can be obtained from C by a slight deformation will serve equally well for the continuation of u into Γ. As each monomial in (2) is analytic almost everywhere, we can take a curve close to C all along which each monomial can be continued from z_0. It is easy to see that a single curve can be taken for all the monomials, because a curve which will do for one of them can be shifted slightly so as to do also for another. We conclude that in any area in which u has an analytic branch, there is an area in which all the monomials in (2) have analytic branches which satisfy (2) together with u. Evidently we can choose the smaller area in such a way that each of the algebraic functions of which the monomials in (2) are exponentials or logarithms is analytic in the smaller area.

5. Consider the domain of rationality of all of the monomials in (2). We can form this domain by taking all rational combinations of the given elements of the monomials, with centers at z_0, and continuing the functions thus obtained. If the first member of (2) is reducible in this domain, let it be replaced by that one of its irreducible factors which vanishes for the

given element of u. We may thus assume that the discriminant of (2), which is analytic in any region in which the α's (properly associated branches of them) are analytic, and in which α_0 is not zero, does not vanish for every z. We see now that u is analytic almost everywhere, since in the neighborhood of any curve there is a curve along which each α is analytic, and on which α_0 and the discriminant of (2) are everywhere different from zero.

6. Let the monomials of order one, some exponentials, some logarithms, which appear in (2) be $\theta_1(z), \cdots, \theta_r(z)$. Suppose that, in every α, we replace each θ_i by a variable z_i'. We form thus an equation

$$\alpha_0(z; z_i')v^m + \alpha_1(z; z_i')v^{m-1} + \cdots + \alpha_m(z; z_i') = 0.$$

Let a be any value of z at which the monomials are all analytic, and at which α_0 and the discriminant of (2) are not zero. Then for $z = a$, $z_i' = \theta_i(a)$ $(i = 1, \cdots, r)$, the first coefficient of the equation for v, and the discriminant, do not vanish. We obtain thus an algebraic function v of $z, z_1', \cdots, z_r'$, analytic when these variables remain in the neighborhood of $z = a$, $z_i' = \theta_i(a)$, and which, when each z_i' is replaced by $\theta_i(z)$, reduces, for a neighborhood of $z = a$, to the function u defined by (2). We observe that the equation for v is independent of the point a.

7. Comparing § 4 and § 6, we see that if u is a function of order 1, then for any area in which (some branch of) u is analytic, there exist

(0) a point a interior to the area, a $\varrho > 0$ and a $\varrho_1 > \varrho$;

(I) r algebraic functions of z, each analytic for $|z-a| < \varrho_1$;

(I′) r monomials, $\theta_1, \cdots, \theta_r$, each either an exponential or a logarithm of one of the r functions in (I), each analytic for $|z-a| < \varrho$, and such that $|\theta_i(z) - \theta_i(a)| < \varrho_1$ for $|z-a| < \varrho$ $(i = 1, \cdots, r)$;

(II) an algebraic function of the variables $z, z_1', \cdots, z_r'$ which is analytic for $|z-a| < \varrho_1$, $|z_i' - \theta_i(a)| < \varrho_1$ $(i = 1, \cdots, r)$, and which reduces to (the given branch of) u for $|z-a| < \varrho$, if each z is replaced by θ_i.*

Furthermore, the integer r, the algebraic equations satisfied by the functions in (I) and that in (II), and the exponential or logarithmic characters of the θ's, are independent of the area in which u is considered and of the branch of u.

8. We now define, by induction, functions of any order n. The exponential or a logarithm of a function of order $n-1$ will be called a *monomial of order n*, provided that it is not among the functions of orders $0, 1, \cdots, n-1$.

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* The fact that this algebraic function may actually depend on z explains our insistence that ρ_1 exceed ρ.

With the same reservation, any function defined by an equation like (2), in which each α is a rational integral combination of monomials of order $0, 1, \cdots, n$, is a *function of order* n*. As above, we may assume that the discriminant of (2) does not vanish identically. One sees by a quick induction that a function of any order n is analytic almost everywhere.

9. As in § 7, we find by induction that if u is a function of any order n, then, for any area in which some branch of u is analytic, there exist

(0) a point a interior to the area, a $\varrho > 0$ and a $\varrho_1 > \varrho$;

(I) r_1 algebraic functions of z, each analytic for $|z - a| < \varrho_1$;

(I′) r_1 monomials, $\theta_1', \cdots, \theta_{r_1}'$, each either an exponential or a logarithm of one of the functions in (I), each analytic for $|z-a| < \varrho$, and such that $|\theta_i'(z) - \theta_i'(a)| < \varrho_1$ for $|z - a| < \varrho$ and for every i;

(II) r_2 algebraic functions of z and of r_1 other variables $z_1', \cdots, z_{r_1}'$, each analytic for $|z - a| < \varrho_1$, $|z_i' - \theta_i'(a)| < \varrho_1$;

(II′) r_2 monomials, $\theta_1'', \cdots, \theta_{r_2}''$, each either an exponential or a logarithm of one of the functions of order 1 to which the algebraic functions in (II) reduce when each z_i' is replaced by θ_i'; each θ_i'' is analytic for $|z - a| < \varrho$, and also $|\theta_i''(z) - \theta_i''(a)| < \varrho_1$ for $|z - a| < \varrho$;

(III) r_3 algebraic functions of $z, z_1', \cdots, z_{r_1}'$ and of r_2 variables $z_1'', \cdots, z_{r_2}''$, each analytic for $|z - a| < \varrho_1$, $|z_i' - \theta_i'(a)| < \varrho_1$, $|z_i'' - \theta_i''(a)| < \varrho_1$;

. . . .

(N+I) an algebraic function of $z; \cdots; z_1^{(n)}, \cdots, z_{r_n}^{(n)}$, analytic for $|z - a| < \varrho_1, \cdots, |z_i^{(n)} - \theta_i^{(n)}(a)| < \varrho_1$, which reduces to the given branch of u for $|z - a| < \varrho$, when each variable z is replaced by the monomial which corresponds to it.

Furthermore the integers r_i, the algebraic equations satisfied by the functions in (I), $\cdots$, (N+I), and the character of the θ's as exponentials or logarithms are independent of the areas in which u is considered, and of the branch of u.

We see that an accented z may be used in forming a monomial of higher order than that to which it corresponds, and be used again by itself.† We have chosen a symbolism which allows this, for the purposes of § 11.

10. For any n, the functions of orders $0, 1, \cdots, n$ form a set which is closed with respect to all algebraic operations. That is, a function defined by an equation like (2), in which each α is a rational integral combination of *functions* of orders $0, 1, \cdots, n$, is itself a function of one of those orders. This follows immediately from (N+I) of § 9, if one considers that an algebraic function of algebraic functions is also algebraic.

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* The existence of functions of all orders is proved by Liouville.

† Consider $\log (e^z + 1) + e^z$.

The functions to which orders are assigned by the preceding definitions will be called *elementary functions* of z.

11. We consider now the differentiation of the elementary functions. Of the algebraic functions introduced in (I), $\cdots$, (N) of § 9, there are possibly some which are used for forming logarithmic monomials. As each monomial is analytic at a, such an algebraic function cannot vanish when z is a, and each accented z is its $\theta(a)$; the function is therefore distinct from zero if the z's are close to these values. If now ϱ_1 is taken sufficiently small, and if ϱ is made correspondingly small, so as to limit the variation of the monomials, we may assume that none of the algebraic functions which give logarithmic monomials vanish when z differs from a, and each accented z from its $\theta(a)$, by an amount smaller than ϱ_1 in modulus.

This understood, the formulas for the differentiation of composite functions show that *if u is an elementary function, described as in* (N+I) *of* § 9, *there exists an algebraic function of the z's, analytic for* $|z-a|<\varrho_1$, $\cdots$, $|z_i^{(n)}-\theta_i^{(n)}(a)|<\varrho_1$, *which reduces to the derivative of u for* $|z-a|<\varrho$, *when each variable is replaced by the monomial which corresponds to it.* A similar result holds for the higher derivatives of u.

12. The equation (2) which defines a function u of order n is never unique, except for $n=0$. But of all the equations (2) which determine u, there are some which involve a minimum number of monomials of order n; that is, the r_n in (N + I) of § 9 is a minimum. In that case, no algebraic relation can exist between these r_n monomials of order n and monomials of order less than n. We mean by this that if $\xi_1, \cdots, \xi_p$ are monomials of order less than n, analytic at $z=a$, and if a function

$$f(z_1^{(n)}, \cdots, z_{r_n}^{(n)};\ x_1, \cdots, x_p),$$

algebraic in all its variables, and analytic for $z_i^{(n)}=\theta_i^{(n)}(a)$, $x_i=\xi_i(a)$, should vanish for the neighborhood of a when each $z_i^{(n)}$ is replaced by $\theta_i^{(n)}$ and each x_i by ξ_i, then the function vanishes for any $z^{(n)}$'s close to the values $\theta^{(n)}(a)$, if only each x_i is replaced by ξ_i.

For suppose that this is not so. Then there is a point b, close to a, such that for $x_i=\xi_i(b)$ $(i=1, 2, \cdots, p)$, and for certain values of the $z^{(n)}$'s close to the $\theta^{(n)}(a)$'s, f does not vanish. Consider the partial derivatives of f, of all orders, with respect to the $z^{(n)}$'s.* Not all of them can vanish for $x_i=\xi_i(b)$, $z_i^{(n)}=\theta_i^{(n)}(b)$, else we could not make f different from zero by varying the $z^{(n)}$'s slightly from the $\theta^{(n)}(b)$'s. (Each $\theta^{(n)}(b)$ is close to $\theta^{(n)}(a)$.)

* Cross-derivatives included.

Suppose then that all of the derivatives up to and including those of order j vanish over the neighborhood of a when the variables are replaced by their monomials, but that some derivative of order $j+1$ does not vanish for a b close to a. To fix our ideas, suppose that

$$g(z_1^{(n)}, \cdots, z_{r_n}^{(n)}; x_1, \cdots, x_p)$$

is a partial derivative which vanishes over the neighborhood of a, but that the derivative of g with respect to $z_1^{(n)}$ does not vanish at b. Then the equation $g=0$ determines $z_1^{(n)}$ as an algebraic function of $z_2^{(n)}, \cdots, x_p$, analytic in the neighborhood of $\theta_2^{(n)}(b), \cdots, \xi_p(b)$, which reduces to $\theta_1^{(n)}$ for the familiar replacements. If we substitute this algebraic function for $z_1^{(n)}$ in $(\mathrm{N}+\mathrm{I})$ of § 9, we find a contradiction of the assumption that r_n is a minimum.

The foregoing principle is due to Liouville, and underlies all of his work on the elementary functions.

II. Some lemmas

13. By a *logarithmic sum of order* n, we shall mean a function of order n of the form

$$c_1 \log \varphi_1(z) + \cdots + c_m \log \varphi_m(z) \qquad (m \geq 1),$$

where each c is a constant, and each $\varphi(z)$ a function of order not exceeding $n-1$. Of course, at least one $\varphi(z)$ is of order $n-1$.*

If we assume that m is a minimum, it follows that no relation $\sum p_i c_i = 0$ can exist with the p's integral and not all zero. For if, for instance, $c_m = \sum_{i=1}^{m-1} q_i c_i$, with the q's rational, the sum could be written $\sum_{i=1}^{m-1} c_i \log \varphi_i \varphi_m^{q}$.

A function defined by an equation (2) in which each a is a rational integral combination of exponential monomials of order n, of logarithmic sums of order n and of monomials of order less than n, will be of order n or less. We may reword $(\mathrm{N}+\mathrm{I})$ of § 9, (also (N) and (N′)), so as to permit the substitution of logarithmic sums of order n, with any number of terms, for some of the variables $z^{(n)}$.† The results of §§ 11, 12 evidently hold for this new type of substitution.

14. The proof of the following lemma wili sharpen its statement.

Lemma. *If, in the expression for a function u of order n, the number of exponentials of order n plus the number of logarithmic sums of order n*

* The present investigation seems to be the first in which sums of logarithms play the rôle of monomials.

† For the $z^{(p)}$ with $p < n$, we shall continue to substitute only monomials.

is a minimum, each exponential of order n and each logarithmic sum of order n is an algebraic function of u, a certain number of the derivatives of u and the monomials of order less than n which appear in the expression for u.

We represent the derivatives of u by u', u'', etc. According to § 11, there exists an infinite sequence of algebraic functions

$$
\begin{aligned}
v &= f_0(z_1^{(n)}, \cdots, z_{r_n}^{(n)}; z_1^{(n-1)}, \cdots, z),\\
v' &= f_1(z_1^{(n)}, \cdots, z_{r_n}^{(n)}; z_1^{(n-1)}, \cdots, z),\\
v'' &= f_2(z_1^{(n)}, \cdots, z_{r_n}^{(n)}; z_1^{(n-1)}, \cdots, z),\\
&\cdots\cdots\cdots\cdots\cdots
\end{aligned}
\tag{3}
$$

which reduce respectively to u, u', u'', etc., for the neighborhood of $z = a$ ((I) of § 9), when each z is replaced by its corresponding monomial or logarithmic sum. The functions of (3) are analytic when the variables are close to the values which their corresponding monomials or sums assume at $z = a$.

Consider any r_n functions of (3). The functional determinant of these functions with respect to $z_1^{(n)}, \cdots, z_{r_n}^{(n)}$ is algebraic in all the z's. If, for some b close to a, this jacobian does not vanish when the z's are replaced by the values which their monomials or sums assume at b, we can solve for the $z^{(n)}$'s; each $z^{(n)}$ will be an algebraic function of $z_1^{(n-1)}, \cdots, z$ and a certain number of v's, which reduces to the exponential or logarithmic sum corresponding to that $z^{(n)}$ when $z_1^{(n-1)}, \cdots, z$ are properly replaced and when each $v^{(i)}$ is replaced by $u^{(i)}$.* This is the state of affairs sought in the lemma.

We are going to show that, because r_n is a minimum, there must be r_n functions in (3) whose jacobian does not vanish throughout the neighborhood of a.

Let the contrary be assumed. We observe first that the derivative of v with respect to $z_1^{(n)}$ cannot vanish for every z close to a. If it did, then, according to Liouville's principle, it would vanish for any $z_1^{(n)}$ close to $\theta_1^{(n)}(a)$, if only the other z's are replaced by their monomials or logarithmic sums. This would mean that u is obtained from

$$f(\theta_1^{(n)}(a), z_2^{(n)}, \cdots, z_{r_n}^{(n)}; z_1^{(n-1)}, \cdots, z)$$

by the familiar replacements, and that r_n is not a minimum.

* $v^{(0)} = v$, $u^{(0)} = u$.

Suppose then that for certain $m < r_n$ of the functions (3), the jacobian with respect to m of the $z^{(n)}$'s, say $z_1^{(n)}, \cdots, z_m^{(n)}$, does not vanish for some b close to a, but that the jacobian of any $m+1$ functions of (3) with respect to any $m+1$ of the $z^{(n)}$'s vanishes for every z close to a. Let the m functions be

$$
(4)\qquad \begin{aligned}
v^{(i_1)} &= f_{i_1}(z_1^{(n)}, \cdots, z_m^{(n)}; z_{m+1}^{(n)}, \cdots, z_{r_n}^{(n)}; z_1^{(n-1)}, \cdots, z),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
v^{(i_m)} &= f_{i_m}(z_1^{(n)}, \cdots, z_m^{(n)}; z_{m+1}^{(n)}, \cdots, z_{r_n}^{(n)}; z_1^{(n-1)}, \cdots, z).
\end{aligned}
$$

In (4), let $z_1^{(n-1)}, \cdots, z$ be replaced by the values of their monomials at $z = b$, and each $v^{(p)}$ by the value of $u^{(p)}$ at $z = b$. Then $z_1^{(n)}, \cdots, z_m^{(n)}$ become functions of $z_{m+1}^{(n)}, \cdots, z_{r_n}^{(n)}$, analytic when the latter variables stay in the neighborhood of $\theta_{m+1}^{(n)}(b), \cdots, \theta_{r_n}^{(n)}(b)$. We are going to allow $z_{m+1}^{(n)}, \cdots, z_{r_n}^{(n)}$ to vary in the following way as functions of a parameter μ. Suppose that $\theta_{m+1}^{(n)}, \cdots, \theta_p^{(n)}$ are exponentials, while $\theta_{p+1}^{(n)}, \cdots, \theta_{r_n}^{(n)}$ are logarithmic sums. We put

$$
(5)\qquad \begin{aligned}
z_{m+1}^{(n)} &= (1+\mu)\,\theta_{m+1}^{(n)}(b), \cdots, z_p^{(n)} = (1+\mu)\,\theta_p^{(n)}(b),\\
z_{p+1}^{(n)} &= \theta_{p+1}^{(n)}(b) + \mu, \qquad \cdots, z_{r_n}^{(n)} = \theta_{r_n}^{(n)}(b) + \mu.
\end{aligned}
$$

Then $z_1^{(n)}, \cdots, z_m^{(n)}$ become functions of μ, analytic for $\mu = 0$. Suppose that $\theta_1^{(n)}, \cdots, \theta_l^{(n)}$ are exponentials, while $\theta_{l+1}^{(n)}, \cdots, \theta_m^{(n)}$ are logarithmic sums. We define functions $\beta(\mu)$, analytic at $\mu = 0$, by

$$
(6)\qquad \begin{aligned}
z_1^{(n)} &= \beta_1(\mu)\,\theta_1^{(n)}(b), \qquad \cdots, z_l^{(n)} = \beta_l(\mu)\,\theta_l^{(n)}(\mu),\\
z_{l+1}^{(n)} &= \theta_{l+1}^{(n)}(b) + \beta_{l+1}(\mu), \cdots, z_m^{(n)} = \theta_m^{(n)}(b) + \beta_m(\mu).^*
\end{aligned}
$$

Thus if all the $z^{(n)}$'s in (4) are replaced by the functions of μ associated with them, the other z's by the values of their monomials at b, the second members in (4) stay constant as μ ranges over the neighborhood of zero.

Consider now any $v^{(q)}$ of (3) where q is distinct from every i of (4). The jacobian of $v^{(q)}$ and of the functions of (4) with respect to any $m+1$ of the $z^{(n)}$'s vanishes for the neighborhood of a. By Liouville's principle, such a jacobian must vanish for arbitrary $z^{(n)}$'s close to their respective $\theta^{(n)}(b)$'s if only the other z's are replaced by their monomials and sums.

* $\beta_i(0)$ is zero or one according as i does or does not exceed l.

It follows from well known theorems on functional dependence that if, in $v^{(q)}$, $z^{(n-1)}, \cdots, z$ are replaced by the values of their monomials at b and if $z_1^{(n)}, \cdots, z_{r_n}^{(n)}$ vary, according to (5) and (6) for instance, so as to keep the functions in (4) constant, $v^{(q)}$ will also stay constant.

The function v' of (3) is derived from v by a formula

$$v' = \sum \frac{\partial v}{\partial z_i^{(n)}} z_i^{(n)} \varphi_i + \sum \frac{\partial v}{\partial z_j^{(n)}} \varphi_j + \text{other terms},$$

where the $z_i^{(n)}$'s correspond to exponentials and the $z_j^{(n)}$'s to logarithmic sums. Each φ_i is a function of $z_1^{(n-1)}, \cdots, z$ which reduces to the derivative of the exponent in $\theta_i^{(n)}$ for the proper replacements; each φ_j is a function of $z^{(n-1)}, \cdots, z$ which reduces to the derivative of $\theta_j^{(n)}$. The "other terms" are derivatives of v with respect to $z_1^{(n-1)}, \cdots, z$ times algebraic functions which reduce to the derivatives of $\theta_1^{(n-1)}, \cdots, z$. It follows that if each $z_i^{(n)}$ is replaced in v' by $k_i \theta_i^{(n)}$ (k_i a constant close to unity) and each $z_j^{(n)}$ by $\theta_j^{(n)} + k_j$ (k_j a constant close to zero), and $z^{(n-1)}, \cdots, z$ by their monomials, the function obtained is the derivative of the function obtained from v by these same replacements. Similarly, v'' etc. will give the higher derivatives of the new function obtained from v.

If, in (5) and (6), we write z in place of b, the $z^{(n)}$'s are associated with functions of z and μ, analytic for $z = b$, $\mu = 0$. If, in v, we replace the $z^{(n)}$'s by these functions, and the other z's by their monomials, we obtain, for any μ, a function u_μ of z. By what we have just seen, the derivatives of u_μ with respect to z for $z = b$ are obtained by making the substitutions (5) and (6), and replacing $z_1^{(n-1)}, \cdots, z$ by $\theta_1^{(n-1)}(b), \cdots, b$ in the functions of (3). Thus the discussion of $v^{(q)}$ above shows that

$$u_\mu(b) = u(b), \qquad u'_\mu(b) = u'(b), \qquad u''_\mu(b) = u''(b), \cdots.$$

Hence, as u_μ and u are analytic in z, *they are identical.*

Thus the partial derivative of u_μ with respect to μ is zero for every admissible z and μ. We equate to zero this partial derivative for $\mu = 0$, and find, using (5) and (6) with b replaced by z,

$$(7) \quad \sum_{i=1}^{l} \beta_i'(0) \frac{\partial v}{\partial z_i^{(n)}} z_i^{(n)} + \sum_{i=l+1}^{m} \beta_i'(0) \frac{\partial v}{\partial z_i^{(n)}} + \sum_{i=m+1}^{p} \frac{\partial v}{\partial z_i^{(n)}} z_i^{(n)} + \sum_{i=p+1}^{r_n} \frac{\partial v}{\partial z_i^{(n)}} = 0.$$

In (7), each z is to be replaced by its monomial or logarithmic sum. But, according to Liouville's principle, (7) will also hold for arbitrary $z^{(n)}$'s if the other z's are replaced by their monomials.

The fact that some of the coefficients $\beta_i'(0)$ may be zero makes a change of notation desirable. Every $z^{(n)}$ of (7), for which $\beta_i'(0)=0$, we replace by a symbol w_q. Every other $z^{(n)}$ we replace by an x_q or a y_q, according as it corresponds to an exponential or to a logarithmic sum. If there are j of the w's, h of the x's, k of the y's, we have $j+h+k=r_n$. The first function of (3) becomes

$$(8)\qquad v=f(w_1,\cdots,w_j;\ x_1,\cdots,x_h;\ y_1,\cdots,y_k;\ z_1^{(n-1)},\cdots,z),$$

the order of its arguments probably being disturbed, while (7) assumes the form

$$(9)\qquad \gamma_1 x_1\frac{\partial v}{\partial x_1}+\cdots+\gamma_h x_h\frac{\partial v}{\partial x_h}+\delta_1\frac{\partial v}{\partial y_1}+\cdots+\delta_k\frac{\partial v}{\partial y_k}=0.$$

Here each γ or δ is either unity or a $\beta'(0)\neq 0$. Also (9) holds for arbitrary, but admissible, w's, x's and y's if $z_1^{(n-1)},\cdots,z$ are replaced by their monomials.

Suppose first that some x's are actually present in (9). We may, after a division, assume that $\gamma_1=1$. Consider then the following $h+k-1$ solutions of (9):

$$(10)\qquad \begin{aligned} s_2&=x_2x_1^{-\gamma_2}, &&\cdots,\ s_h=x_hx_1^{-\gamma_h},\\ t_1&=y_1-\delta_1\log x_1, &&\cdots,\ t_k=y_k-\delta_k\log x_1.\end{aligned}$$

These solutions are analytic for the values in which we are interested of the x's and the y's, because x_1, which is associated with the exponential of an analytic function, does not become zero. The jacobian of these solutions with respect to $x_2,\cdots,y_k$ is $x^{-(\gamma_2+\cdots+\gamma_h)}$, which is not zero. Consequently if the w's are given arbitrary fixed values, and if $z_1^{(n-1)},\cdots,z$ are held fast at the values of their monomials for any fixed z, v in (8) becomes an analytic function of the functions (10). If we replace $x_2,\cdots,y_k$ by their values obtained from (10), we find

$$(11)\qquad v=f(w_1,\cdots;\ x_1,\ x_1^{\gamma_2}s_2,\cdots;\ t_1+\delta_1\log x_1,\cdots;\ z^{(n-1)},\cdots,z).$$

By what precedes, the second member of (11) is independent of x_1, so that

$$(12)\qquad v=f(w_1,\cdots;\ c_1,c_2s_2,\cdots;\ t_1+d_1,\cdots;\ z_1^{(n-1)},\cdots,z),$$

where the c's and d's are constants.

We notice that when the x's are replaced by their exponentials, each s becomes an exponential of a function which is at most of order $n-1$, and each t a logarithmic sum of order n plus a function of order $n-1$. If then we replace the variables in (12) by the functions of z to which they correspond, we have u expressed in terms of fewer than r_n exponentials and sums of order n. This contradiction of the assumption that r_n is a minimum implies the truth of the lemma.

If no x's are present in (9) $(h=0)$, we use the independent solutions of (9),

$$t_2 = \delta_2 y_2 - \delta_1 y_1, \quad \cdots, \quad t_k = \delta_k y_k - \delta_1 y_1.$$

As above, we find that r_n is no minimum. This completes the proof of the lemma.

15. We shall call any set of numbers, $c_1, \cdots, c_m$ *dependent* or *independent* according as there do or do not exist integers $p_1, \cdots, p_m$, not all zero, such that $\sum p_i c_i = 0$.

LEMMA. *A function* $\sum_{i=1}^{m} c_i \log \varphi_i(z)$, *with no* $\varphi_i(z)$ *of order greater than* $n-1$, *with at least one* $\log \varphi_i(z)$ *of order* n, *and with independent* c's, *is a function of order* n.

We begin by proving the theorem for the case of $n=1$. Suppose then that each $\varphi_i(z)$ is an algebraic function, that some $\varphi_i(z)$ is not constant, but that the sum of logarithms is an algebraic function $\psi(z)$. Differentiating, we have

$$c_1 \frac{\varphi_1'(z)}{\varphi_1(z)} + \cdots + c_m \frac{\varphi_m'(z)}{\varphi_m(z)} = \psi'(z). \tag{13}$$

Suppose that a function $\varphi_i(z)$ has a zero or a pole at some point a, which may or may not be a branch point of the function. Then $\varphi_i'(z)/\varphi_i(z)$ will have a pole at a in which the coefficient of $1/(z-a)$ is a rational number; the coefficient may be a fraction if a is a branch point, but otherwise it is an integer. Thus the first member of (13) has a development at a in which the coefficient of $1/(z-a)$ is a linear combination of the c's with rational coefficients, some coefficients distinct from zero. But we cannot get a term in $1/(z-a)$ by differentiating an algebraic function $\psi(z)$, so that (13) is impossible.

Suppose now that the lemma is untrue for some $n>1$, so that there exists a class of functions $\psi = \sum_{i=1}^{m} c_i \log \varphi_i$ of order less than n, with each φ_i of order less than n, with some $\log \varphi_i$ of order n, and with independent c's. Here m may depend on ψ, but this is not of importance.

For any ψ of this class, let r represent the minimum number of monomials of order $n-1$ in terms of which, with monomials of lower order, ψ and all of the functions φ_i can be expressed. Consider the subclass formed by those functions ψ whose r is not greater than the r of any other ψ. We may assume that the functions of this subclass are so expressed that of the r monomials of order $n-1$ appearing in $\psi, \varphi_1, \cdots, \varphi_m$, the number s of those which appear in $\varphi_1, \cdots, \varphi_m$ is a minimum. In the subclass there are certain functions ψ whose s is not greater than the s of any other function of the subclass. We assume that we have in hand a ψ of this type, and proceed to force a contradiction.

Writing w for $z^{(n-1)}$, we have, for the familiar replacements,

$$\sum_{i=1}^{m} c_i \log f_i(w_1, \cdots, w_s; z^{(n-2)}, \cdots, z) = g(w_1, \cdots, w_r; z^{(n-2)}, \cdots, z), \tag{14}$$

each φ_i resulting from f_i, and ψ from g. After differentiation, (14) gives, for the replacements,

$$\sum_{i=1}^{m} c_i \frac{f_i'}{f_i} = g', \tag{15}$$

where the significance of f_i' and g' is obvious. As usual, (15) holds for arbitrary w's.

We shall prove first that none of $w_1, \cdots, w_s$ can be associated with a logarithm. Suppose, for instance, that w_1 corresponds to a logarithm, θ. As seen in § 15, if, in (14) and (15), w_1 is replaced by $\theta + \mu$ (μ constant and small), the other w's and z's by their monomials, the members of (15) will still be the derivatives of those of (14). Also, (15) will remain an equation. Consequently, for these replacements, we have

$$\sum c_i \log f_i = g + \beta(\mu), \tag{16}$$

where $\beta(\mu)$, being the difference of two analytic functions of μ, is analytic for μ small. We differentiate with respect to μ in (16), and put $\mu = 0$, obtaining

$$\sum c_i \frac{1}{f_i} \frac{\partial f_i}{\partial w_1} = \frac{\partial g}{\partial w_1} + \beta'(0). \tag{17}$$

Again, (17) holds for arbitrary w's, but we consider only w_1 arbitrary, and replace the other w's. Integrating (17), we have, for w_1 arbitrary,

$$\sum c_i \log f_i = g + \beta'(0) w_1 + \gamma(z), \tag{18}$$

where it will be unnecessary to determine $\gamma(z)$.

By what we know for the case of $n = 1$, (18) shows that when $w_2, \cdots, z$ are replaced by their monomials, each $\log f_i$ (and also $g + \beta'(0) w_1$) becomes independent of w_1. But this contradicts the assumption that s is a minimum, so that w_1 cannot stand for a logarithm.

Suppose then that w_1 corresponds to an exponential, θ. We find that (16) holds when w_1 is replaced by $\mu\theta$, with μ close to 1. Differentiating with respect to μ, and putting $\mu = 1$, we have

$$\sum c_i \frac{1}{f_i} \frac{\partial f_i}{\partial w_1} w_1 = \frac{\partial g}{\partial w_1} w_1 + \beta'(1).$$

Letting w_1 be arbitrary, and integrating, we find

$$\sum c_i \log f_i - \beta'(1) \log w_1 = g + \gamma(z). \tag{19}$$

If $\beta'(1)$ were not a linear combination of the c's with rational coefficients we would have, on fixing z, a contradiction. Thus, let

$$\beta'(1) = q_1 c_1 + \cdots + q_m c_m$$

with rational q's. Then (19) gives, for w_1 arbitrary,

$$\sum c_i \log \frac{f_i}{w_1^{q_i}} = g + \gamma(z).$$

Consequently, for every i, $f_i / w_1^{q_i}$ is independent of w_1, and if we write (14)

$$\sum c_i \log \frac{f_i}{w_1^{q_i}} = g - \beta'(1) \log w_1, \tag{20}$$

we may replace w_1 in the first member by a constant instead of by θ.

Now some term in the first member of (20) is of order n, because we have subtracted from each $\log \varphi_i$ a function of order $n-2$. Also the order of the second member is less than n. This contradiction of the assumption that s is a minimum proves the lemma.

16. If $\varphi(z)$ is of order n, $\log \varphi(z)$ may be of any of the orders $n-1$, n, $n+1$. We prove the

LEMMA. *If $\varphi(z)$ and $\log \varphi(z)$ are both of order $n > 0$, $\varphi(z)$ is of the form $\xi_1(z) e^{\xi_2(z)}$, where $\xi_1(z)$ and $\xi_2(z)$ are each of order $n-1$.*

Let $\psi = \log \varphi$. We choose expressions for φ and ψ such that the total number r of monomials of order n appearing in both of them is a minimum,

and this condition being first satisfied, we suppose further that we have expressions such that s, the number of monomials of order n appearing in φ, is a minimum.

We have, for the replacements,

$$(21)\quad \log f(w_1, \cdots, w_s;\ z^{(n-1)}, \cdots, z) = g(w_1, \cdots, w_r;\ z^{(n-1)}, \cdots, z),$$

φ resulting from f and ψ from g.

Precisely as in § 15, we prove that $w_1, \cdots, w_s$ cannot correspond to logarithms. Suppose that w_1 corresponds to an exponential, θ_1. We find the equation

$$\log f = g + \beta(\mu)$$

to hold when w_1 is replaced by $\mu\theta_1$. Then

$$\frac{1}{f}\,\frac{\partial f}{\partial w_1} w_1 = \frac{\partial g}{\partial w_1} w_1 + \beta'(1),$$

so that, for w_1 arbitrary,

$$\log f - \beta'(1)\log w_1 = g + \gamma(z).$$

This means that $\beta'(1)$ is a rational number q_1, and that $f/w_1^{q_1}$ is independent of w_1 when $w_2, \cdots, z$ are replaced. Writing (21)

$$(22)\qquad \log\frac{f}{w_1^{q_1}} = g - q_1\log w_1,$$

replacing w_1 by a constant in the first member and by θ_1 in the second, we have again, if $s > 1$, a function of order n whose logarithm is also of order n. Continuing thus, we find that $\varphi(z)$ divided by $\theta_1^{q_1}\theta_2^{q_2}\cdots\theta_s^{q_s}$ is a function of order $n-1$ at most, and this proves the lemma.

As an immediate consequence of the above result, we have the

LEMMA. *If $\varphi(z)$ and $e^{\varphi(z)}$ are both of order $n > 0$, $\varphi(z) = \xi_1(z) + \log\xi_2(z)$, where $\xi_1(z)$ and $\xi_2(z)$ are each of order $n-1$.*

17. We record here two results, easily proved, of which we shall later use the second.

If $\varphi(z)$ is of order n, and if $\varphi'(z)$ is of order less than n, $\varphi(z) = \varphi_1(z) + \varphi_2(z)$, where $\varphi_1(z)$ is of order less than n, and where $\varphi_2(z)$ is a sum of logarithms of order n multiplied by constants.

If $\varphi(z)$ is of order n, and if the logarithmic derivative of $\varphi(z)$ is of order less than n, $\varphi(z) = \varphi_1(z)e^{\varphi_2(z)}$ where $\varphi_1(z)$ is of order less than n, and where $\varphi_2(z)$ is of order $n-1$.

III. Composite elementary functions

18. In what follows, we shall discontinue the "replacement" language, and speak of the arguments w, z etc. in our algebraic functions as "being" monomials. What precedes indicates sufficiently how everything we say is to be taken.

LEMMA. *Given a function $\varphi(z)$ of order m, if a function $\psi(z)$ of order $n>1$ exists such that the order of $\psi[\varphi(z)]$ does not exceed $m+n-2$, there exists a monomial of order n, $\theta(z)$, such that $\theta[\varphi(z)]$ is at most of order $m+n-2$.*

According to § 14, if w is one of a minimum number of monomials and sums of order n in the expression for $\psi(z)$, we have, with f algebraic,

$$w = f(z_1^{(n-1)}, \cdots, z; \psi, \psi', \cdots, \psi^{(p)}). \tag{23}$$

From § 11, we see that the order of the derivative of a function does not exceed the order of the function. Thus, since

$$\psi'[\varphi(z)] = \frac{1}{\varphi'(z)}\frac{d}{dz}\psi[\varphi(z)],$$

the order of $\psi'[\varphi]$ does not exceed the greater of $m+n-2$ and m. By induction, the order of every $\psi^{(i)}[\varphi]$ is seen not to exceed the greater of these integers. As n is now at least 2, the order of no $\psi^{(i)}[\varphi]$ exceeds $m+n-2$.

Thus, by (23), $w[\varphi]$ is at most of order $m+n-1$. Its order will be even less if no $z^{(n-1)}[\varphi]$ is of order $m+n-1$.

Suppose first that $w = e^u$, where u depends on $z_1^{(n-1)}, \cdots, z$. If the order of $w[\varphi]$ does not exceed $m+n-2$, w is the monomial sought in the lemma. In what follows, we assume the order of $w[\varphi]$ to be $m+n-1$.

If a $z^{(n-1)}[\varphi]$ is of order $m+n-1$, it is a *monomial.** Hence $u[\varphi]$ has an expression in which all monomials of order $m+n-1$, if indeed there be any, are of the form $z^{(n-1)}[\varphi]$. By (23), the same is true of $w[\varphi]$.

We choose expressions for $u[\varphi]$ and $w[\varphi]$ such that the total number r of monomials of order $m+n-1$, all of the form $z^{(n-1)}[\varphi]$, appearing in

* When $n>1$, as the hypothesis stipulates.

both of them is a minimum, and this condition being first satisfied, we suppose further that we have expressions such that s, the number of monomials of order $m+n-1$ in $w[\varphi]$, is a minimum.

Let $W = w[\varphi]$, $U = u[\varphi]$. We write x for the monomials of order $m+n-1$, and omit symbols for monomials of lower order. We have

$$\log W(x_1, \cdots, x_s; z) = U(x_1, \cdots, x_r; z).$$

Precisely as in § 16, we prove that $x_1, \cdots, x_s$ are exponentials, and that

$$W = x_1^{q_1} \cdots x_s^{q_s} V.$$

where the q's are rational, and where V is of order $m+n-2$ at most. Now $x_1^{q_1} \cdots x_s^{q_s}$ is of the form $\zeta[\varphi]$, where ζ is an exponential of order $n-1$. Let $\zeta = e^{u_1}$, where u_1 is of order $n-2$. Then $v = u - u_1$ is of order $n-1$ while its exponential is of order n, and we have

$$e^{v[\varphi]} = V,$$

as the lemma requires.

Suppose now that, in (23), w is a logarithmic sum of order n, and that the order of $w[\varphi]$ is $m+n-1$. We shall later cover the case in which the order is less.

Let $w = \sum c_i \log u_i$, with independent c's, where no u_i is of order greater than $n-1$. We put

$$W = w[\varphi], \qquad U_i = u_i[\varphi],$$

observing that W and each U_i have expressions in which every monomial of order $m+n-1$ is of the form $z^{(n-1)}[\varphi]$. Introducing x's, with r a minimum for W and the U_i's and then s a minimum for W alone, we write

$$W(x_1, \cdots, x_s; z) = \sum c_i \log U_i(x_1, \cdots, x_r; z).$$

We prove quickly that $x_1, \cdots, x_s$ are not exponentials. Let x_1 be a logarithm. We find, for x_1 arbitrary,

$$W = \sum c_i \log U_i + \beta'(0)x_1 + \gamma(z),$$

so that, by § 15, $W - \beta'(0)x_1$, and each U_i, are independent of x. Continuing, we find that W less a linear combination of the x's is independent

of the x's, and is hence of order $m+n-2$ at most. But the linear combination of the x's is of the form $\xi[\varphi]$, where ξ is a logarithmic sum of order $n-1$. Let

$$\xi(z) = \sum d_i \log v_i(z),$$

where no v_i is of order greater than $n-2$. Let $\zeta = w - \xi$, so that

$$\zeta(z) = \sum c_i \log u_i(z) - \sum d_i \log v_i(z). \tag{24}$$

Then ζ is a logarithmic sum of order n, and $\zeta[\varphi]$ is at most of order $m+n-2$.

Of course, it might have been that $w[\varphi]$ above was itself of order not exceeding $m+n-2$. If such be the case, ζ is to stand for w in what follows.

Let ζ be reduced to the form $\sum e_i \log t_i$ with no t_i of order greater than $n-1$, and with independent e's. We put $T_i = t_i[\varphi]$. Then each T_i has an expression in which all monomials of order $m+n-1$, if there are any, are of the form $z^{(n-1)}[\varphi]$. We assume that the T's are so expressed that the total number r of such monomials appearing in all of them is a minimum, and putting $Z = \zeta[\varphi]$, we write

$$\sum e_i \log T_i(x_1, \cdots, x_r; z) = Z. \tag{25}$$

We prove quickly that no x is a logarithm. Let x_1 be an exponential. We find, for x_1 arbitrary,

$$\sum e_i \log T_i = \beta'(1) \log x_1 + \gamma(z).$$

By § 15, we must have $\beta'(1) = \sum q_i e_i$ with rational q's, and each $T_i/x_1^{q_i}$ must be independent of x_1. Now x_1 is of the form $\tau[\varphi]$, where τ is an exponential of order $n-1$. We put

$$T'_i = t'_i[\varphi] = \frac{T_i}{x_1^{q_i}}, \qquad Z' = \zeta'[\varphi] = Z - \beta'(1)\log x_1.$$

Then, since $\log \tau$ is of order $n-2$, $\zeta' = \sum e_i \log t'_i$ is a logarithmic sum of order n. Also $\zeta'[\varphi]$ is at most of order $m+n-2$. Finally each $t'_i[\varphi]$ involves only $x_2, \cdots, x_r$ and not x_1.

It is evident that if this process is gone through r times, we will arrive at a logarithmic sum of order n, $\zeta^{(r)} = \sum e_i \log t_i^{(r)}$, such that $\zeta^{(r)}[\varphi]$ and

also each $t^{(r)}[\varphi]$ are at most of order $m+n-2$. It follows by § 15 that no $\log t_i^{(r)}[\varphi]$ has an order greater than $m+n-2$. Since some $\log t_i^{(r)}$ is of order n, the lemma is proved.

The fact that the order of $t_i^{(r)}[\varphi]$ does not exceed $m+n-2$ will be used in the next section.

19. LEMMA. *Given a function $\varphi(z)$ of order m, if a $\psi(z)$ of order $n>1$ exists such that the order of $\psi[\varphi(z)]$ does not exceed $m+n-2$, there exists a function $\psi_1(z)$ of order $n-1$, where either $\log\psi_1(z)$ or $e^{\psi_1(z)}$ is of order n, such that the order of $\psi_1[\varphi(z)]$ does not exceed $m+n-2$.*

According to the preceding section, we may assume that ψ is a monomial, and indeed, the final remark of that section disposes of the case in which ψ is a logarithm.

Suppose that ψ is an exponential e^w. We have to discuss the case in which $w[\varphi]$ is of order $m+n-1$. Of course, $w[\varphi]$ has an expression in which every monomial of order $m+n-1$ is of the form $z^{(n-1)}[\varphi]$. This is because $n>1$. Let $W=w[\varphi]$, and suppose that W is expressed in terms of a minimum number r of monomials of order $m+n-1$, all of the form $z^{(n-1)}[\varphi]$. Writing

$$W(x_1,\cdots,x_r;\,z)=\log\psi[\varphi],$$

we prove that the x's are logarithms, and that $W=\sum c_i x_i+\xi$, where ξ is at most of order $m+n-2$. Here the c's are independent, since r is a minimum. Let $x_i=\log v_i$, where v_i is of order $n-2$. We have

$$c_1\log v_1[\varphi]+\cdots+c_r\log v_r[\varphi]-\log\psi[\varphi]=-\xi.$$

Hence, by § 15, we must have $1=\sum q_i c_i$, with rational q's, so that

$$c_1\log\frac{v_1[\varphi]}{(\psi[\varphi])^{q_1}}+\cdots+c_r\log\frac{v_r[\varphi]}{(\psi[\varphi])^{q_r}}=-\xi.$$

Furthermore, by § 15, no logarithms in the equation just written can be of order greater than $m+n-2$. Thus, considering the first term, we see that

$$q_1\log\psi[\varphi]-\log v_1[\varphi]=q_1 w[\varphi]-\log v_1[\varphi]$$

is of order $m+n-2$ at most.

Hence $q_1 w-\log v_1$ is the function we seek, unless its order is less than $n-1$. But then

$$e^{q_1 w}=v_1 e^{q_1 w-\log v_1}$$

would be of order less than n. As e^w is of order n, and as q_1 is rational, and clearly not zero, this is impossible. The lemma is proved.

20. LEMMA. *Given a function $\varphi(z)$ of order m, if a $\psi(z)$ of order $n>2$ exists such that the order of $\psi[\varphi(z)]$ does not exceed $m+n-2$, there exists a function $\psi_1(z)$ of order $n-1$ such that the order of $\psi_1[\varphi(z)]$ does not exceed $m+n-3$.*

According to §§ 18, 19, we may assume that ψ is a monomial, and that if w is the function whose exponential or logarithm is taken, $w[\varphi]$ is at most of order $m+n-2$.

First let $\psi=e^w$, and suppose that $w[\varphi]$ is of order $m+n-2$. According to § 16, if $\psi[\varphi]$ is of order $m+n-2$, we have

$$w[\varphi]=\xi_1+\log\xi_2 \tag{26}$$

where ξ_1 and ξ_2 are of order $m+n-3$. If $\psi[\varphi]$ is of order $m+n-3$, $\xi_1=0$ in (26).

Let x be one of a minimum number of exponentials and *logarithmic sums* of order $n-1$ in w. Then x is algebraic in w, w', w'', etc., and in monomials of order less than $n-1$. Thus, as $n>2>1$, $x[\varphi]$ is at most of order $m+n-2$; suppose it is actually of order $m+n-2$.

First let x be an exponential, e^v. If $v[\varphi]$ is of order $m+n-3$, $x[\varphi]$ is an exponential of order $m+n-2$. If $v[\varphi]$ is of order $m+n-2$, then, by § 16. $x[\varphi]=\zeta_1 e^{\zeta_2}$ with ζ_1 and ζ_2 of order $m+n-3$.

Again, let x be a logarithmic sum, $\sum c_i\log v_i$, with independent c's. According to § 15, no $\log v_i[\varphi]$ can be of order $m+n-1$. Let $\log v_i[\varphi]$ be of order $m+n-2$. If $v_i[\varphi]$ is of order $m+n-3$, $\log v_i[\varphi]$ is a logarithmic monomial of order $m+n-2$. Otherwise $\log v_i[\varphi]=\zeta_1+\log\zeta_2$, with ζ_1 and ζ_2 of order $m+n-3$.

If $z^{(n-2)}$ is a monomial of order $n-2$ in the expression for w, and if $z^{(n-2)}[\varphi]$ is of order $m+n-2$, then, since $n>2$, $z^{(n-2)}[\varphi]$ is a *monomial* of order $m+n-2$.

In all, we see that $w[\varphi]$ has an expression in which every monomial of order $m+n-2$ is either the product or the sum of a function of order $m+n-3$ at most, and a function $\tau[\varphi]$, where τ is a monomial of order $n-1$ or $n-2$. It is the product if it is an exponential, the sum if a logarithm. Also, the monomial of order $m+n-2$ and τ are either both exponentials or both logarithms.

Of all expressions for $w[\varphi]$ in which the monomials of order $m+n-2$, $y_1,\cdots,y_r$, are of the rather complicated type just described, consider one for which r is a minimum. We prove quickly, using (26), that each y is a logarithm, and that

(27) $$w[\varphi] = \sigma + c_1 y_1 + \cdots + c_r y_r,$$

where the order of σ does not exceed $m+n-3$.* An easy discussion would show that because r is a minimum, the c's are independent, but we get along more simply as follows. We note that each y is a logarithmic monomial of order $m+n-2$, and differs by a function of order less than $m+n-2$ from a function $\tau[\varphi]$, where τ is a logarithm of a function of order not exceeding $n-2$. The fact that τ is a monomial, we do not stress. Of all the representations of $w[\varphi]$ of the form (27) with y's of this type, we take one for which r is a minimum. In that case the c's are evidently independent.

Using (26) and § 15, and the representation just obtained for $w[\varphi]$, we prove that a rational q_1 exists such that y_1 and $q_1 \log \xi_2$ differ by a function of order $m+n-3$ at most. Hence, remembering that y is of the form $\log v[\varphi] - \zeta$ where v is of order $n-2$ or $n-3$, and ζ is of order less than $m+n-2$, we find by (26) that $q_1 w[\varphi] - \log v[\varphi]$ is at most of order $m+n-3$. Thus $q_1 w - \log v$ is the function sought in our lemma, unless its order is less than $n-1$. This is seen, as in § 19, to be impossible.

We take the case in which $\psi = \log w$, with $w[\varphi]$ at most of order $m+n-2$. We could use a discussion similar to that for the exponential case, but the following method is shorter.

If $w[\varphi]$ is of order $m+n-2$, it is of the form e^{ξ_2} or $\xi_1 e^{\xi_2}$, (ξ_1 and ξ_2 of order $m+n-3$), according as $\log w[\varphi]$ is of order $m+n-3$ or $m+n-2$. In any case, the logarithmic derivative of $w[\varphi]$ is of order $m+n-3$ at most. Hence, as $n > 2$, the logarithmic derivative of w is the function sought, unless it is of order less than $n-1$. In the latter case, according to § 17, we would have $w = \zeta_1 e^{\zeta_2}$, where ζ_2 is of order $n-2$, and ζ_1 of order less than $n-1$, so that $\log w$ could not be of order n.

This completes the proof of the lemma.

21. Lemma. *Given a $\varphi(z)$ of order $m > 0$, if a $\psi(z)$ of order two exists such that $\psi[\varphi(z)]$ is at most of order m, then a $\psi_1(z)$ of order one exists such that $\psi_1[\varphi(z)]$ is at most of order $m-1$.*

According to §§ 18, 19, we may assume that $\psi(z)$ is a monomial e^w or $\log w$, with $w[\varphi]$ at most of order m.

Let θ be one of a minimum number of exponentials and sums in w. Then θ is algebraic in z, w, etc., so that $\theta[\varphi]$ is at most of order m. If θ is a logarithmic sum with independent c's, none of the terms in it can become of order greater than m when z is replaced by $\varphi(z)$.

* Liouville's principle applies as usual.

If, when we substitute $\varphi(z)$ into one of the monomials in w, we obtain a function of order $m-1$, the monomial is the function sought in the lemma.

Suppose that this is not so. Then if one of the monomials θ is an exponential, φ is an algebraic function of $\xi_1+\log\xi_2$ with ξ_1 and ξ_2 of order $m-1$, whereas if θ is a logarithm, φ is an algebraic function of $\xi_1 e^{\xi_2}$.* But φ cannot have both forms, so that w cannot contain both logarithms and exponentials.

Suppose first that all monomials are exponentials. If an algebraic function of $\xi_1+\log\xi_2$ (as above) is also of the form $\xi_1+\log\xi_2$, the algebraic function is of the form $az+b$, with a rational. Hence w can contain only one exponential, essentially.

Similarly, w cannot contain more than one logarithm.

The results just obtained are a consequence of the mere fact that the order of $w[\varphi]$ does not exceed m. This will be made use of in the following section.

Consider the case of $\psi = e^w$.

Let, then, $w = f(\theta, z)$, and suppose first that θ is an exponential. If $w[\varphi]$ is of order m, it is of the form $\log\xi$ or $\xi_1+\log\xi_2$, with ξ's of order $m-1$, because $e^{w[\varphi]}$ is at most of order m. Then $\theta[\varphi]$, which has to be of the form $\xi_1 e^{\xi_2}$, cannot be so, for it is algebraic in $w[\varphi]$ and φ. Thus $w[\varphi]$ is at most of order $m-1$.

Thus, if $w[\varphi]$ is of order m, θ cannot be an exponential. If θ is a logarithm, we find quickly that $f(\theta, z) = a\theta + b$, with a rational, so that e^w is not of order 2.

Hence, when ψ is an exponential, $w[\varphi]$ is of order $m-1$ at most. The logarithmic case goes through with only slight changes.

22. LEMMA. *If $\varphi(z)$ is of order $m>0$, and if a $\psi(z)$ of order one exists such that $\psi[\varphi(z)]$ is of order not exceeding $m-1$, then $\varphi(z)$ is an algebraic function of a monomial of order m.*

As noted in § 21, the fact that $\psi[\varphi]$ is of order not exceeding m implies either that ψ has a monomial θ such that $\theta[\varphi]$ is of order $m-1$, or else that ψ is of the form $f(\theta, z)$. In the former case, we have what the lemma requires. In the second case, $\theta[\varphi]$ is algebraic in $\psi[\varphi]$ and φ, so that, by arguments like those of § 21, $\theta[\varphi]$ cannot be of order m. This settles the lemma.

23. Comparing the lemmas of §§ 20—22, we find the

THEOREM. *Given a function $\varphi(z)$ of order m, if a function $\psi(z)$ of order $n>0$ exists such that $\psi[\varphi(z)]$ is at most of order $m+n-2$, then $\varphi(z)$ is an algebraic function of a monomial of order m.*

* The hypothesis prevents φ from being algebraic.

This theorem permits us to determine all elementary functions with elementary inverses. For if $\varphi(z)$ is such a function, of order $m>0$, since $\varphi^{-1}\varphi(z)$ is of order zero, $\varphi(z)$ is an algebraic function of a monomial of order m. But the function of order $m-1$ of which the monomial is an exponential or a logarithm also has an elementary inverse, and is thus algebraic in a monomial of order $m-1$. Continuing thus, we find the result stated in the introduction.

With a set of lemmas only slightly different from those above (the changes are all simplifications), we obtain the

THEOREM. *Given a function $\varphi(z)$ of order $m>0$, if a function $\psi(z)$ of order $n>0$ exists such that $\psi[\varphi(z)]$ is precisely of order $m+n-1$, then $\varphi(z)$ is an algebraic function of a function of one of the forms $\xi_1(z)+\log\xi_2(z)$ or $\xi_1(z)e^{\xi_2(z)}$, where $\xi_1(z)$ and $\xi_2(z)$ are of order $m-1$.*

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ANALYTIC TRANSFORMATIONS OF EVERYWHERE DENSE POINT SETS*

BY

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I. Transformations of point sets

There is a well known theorem in the theory of point sets, due to Cantor,† to the effect that *All enumerable, everywhere dense linear point sets without first and last points have the same order type as the rational numbers.* That is, any set of this type can be mapped on the rational points of a line by a one to one correspondence which preserves order, and consequently any two sets of this type can be mapped on one another by such a correspondence.

A correspondence of two everywhere dense point sets clearly determines at most one continuous function which maps the segments on which the given sets are everywhere dense on one another, and also generates the correspondence. The requirement that the correspondence preserve order is equivalent to the requirement that a continuous mapping function exist, so that we may state the above theorem in the following form: *For any two enumerable linear point sets, each everywhere dense on an open interval, a continuous function can be found which maps the two intervals on one another, and effects a one to one correspondence between the point sets.*

Since the function of this theorem is by no means uniquely determined, the question naturally arises as to whether we can place further restrictions on it without destroying the validity of the theorem. It turns out that we may always require the function which effects the mapping to be *analytic* and it is the demonstration of this fact and some related questions which occupy our attention in this paper.

II. Existence of an analytic transformation

In proving the existence of an analytic mapping function, there is obviously no loss of generality in restricting the two given point sets to lie on the interval from 0 to 1. For a set on the interval a to b is mapped on this unit interval by the transformation $w = (x-a)/(b-a)$, one on the interval a to ∞ by the transformation $w = (x-a)/(1+x-a)$, and

* Presented to the Society, May 3, 1924.

† Mathematische Annalen, vol. 46 (1895), p. 505.

one on the entire straight line by the transformation $w = e^x/(1+e^x)$. Consequently, if we show that two sets of the specified type on the unit interval may always be mapped on one another by an analytic transformation, the combination of one of the three transformations just given, the transformation for the two unit intervals, and the inverse of one of the three will yield an analytic transformation for any two intervals.

Consider then two point sets, each of which is enumerable and everywhere dense on the unit interval. Since the sets are enumerable, we may designate the points of the first set as

$$a_1,\ a_2.\ a_3,\ \cdots$$

and those of the second as

$$b_1,\ b_2,\ b_3,\ \cdots.$$

We shall make use of a set of small positive constants

$$\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3,\ \cdots.$$

selected so that their sum converges:

$$\varepsilon_1+\varepsilon_2+\varepsilon_3+\cdots = h \qquad (h<1)$$

but otherwise arbitrary.

Our method of setting up the mapping function will be one of successive approximation, each new approximation making our function behave properly at a new point, without affecting its behavior at points already considered. We start then with the function

$$y_1 = x,$$

which takes the end points of the two intervals into one another. This function takes the point a_1 into a_1. In general, this is not a b_i. Since, however, the b_i are everywhere dense on the unit interval, we may find a b_i as close to the point a_1 as we please, in particular, a b_{i_1} such that

$$b_{i_1} - a_1 = e_1 \frac{a_1(a_1-1)}{2}; \qquad |e_1| < \varepsilon_1.$$

From this we form our second approximation,

$$y_2 = x + e_1 \frac{x(x-1)}{2}.$$

It will be noted that this takes the unit interval into itself, in a one to one manner (since it has a positive derivative in this interval), and also takes a_1 into b_{i_1}.

If now we give y_2 the value b_1 (or b_2, if $b_{i_1} = b_1$), it will correspond to a single value of x in the unit interval, say x_1, which in general is not an a_i. But, in virtue of the fact that the a_i are everywhere dense, we may find an a_i distinct from a_1 as close to x_1 as we please, in particular, since y_2 is a continuous function, an a_{j_1} such that

$$y_2(a_{j_1}) - b_1 = k_1 \frac{b_1(b_1-1)(b_1-b_{i_1})}{3}; \qquad |k_1| < \epsilon_1.$$

This enables us to form our third approximation,

$$y_3 + k_1 \frac{y_3(y_3-1)(y_3-b_{i_1})}{3} = x + e_1 \frac{x(x-1)}{2}.$$

As the derivative of the left member with respect to y_3 is positive in the unit interval, while that of the right member with respect to x is also positive in this interval, the function $y_3(x)$ maps the unit interval on itself in a one to one manner; it also obviously takes a_1 into b_{i_1} and a_{j_1} into b_1.

We next add a term to the right member to make a_2 (or a_3 if $a_{j_1} = a_2$) correspond to a b_i, say b_{i_2}:

$$e_2 \frac{x(x-1)(x-a_1)(x-a_{j_1})}{4}, \qquad |e_2| < \epsilon_2.$$

Then we add a small term to the left member to make b_2 (or the b with smallest index not already used) correspond to an a_i, say a_{j_2}:

$$k_2 \frac{y(y-1)(y-b_{i_1})(y-b_1)(y-b_{i_2})}{5}, \qquad |k_2| < \epsilon_2.$$

The method of procedure is now clear. At each stage we take the next a_i or b_i as the case may be, which has not been already used, and so change the corresponding member of the approximating equation that it shall correspond to a point of the other set for the new function. The changed term is in the form of a polynomial which vanishes at all the points already adjusted, and a numerical factor is inserted to make it, as well as its derivative, less in absolute value than the corresponding ϵ_n

throughout the interval 0 to 1. The process determines an equation each of whose sides is an infinite series:

$$y+\sum_{1}^{\infty} k_n \frac{y(y-1)(y-b_{i_1})\cdots(y-b_{i_n})}{2n+1} = x+\sum_{1}^{\infty} e_n \frac{x(x-1)(x-a_1)\cdots(x-a_{j_n})}{2n+2}.$$

Let us consider these two series in turn. In the interval 0 to 1, each term in the right member is an analytic function of x, whose absolute value is less than the corresponding ϵ_n. Consequently, since the series of ϵ_n's converges, the right member represents an analytic function of x. Furthermore, since the series obtained by termwise differentiation also is dominated by the ϵ series, it represents the derivative of the function just obtained. As the sum of the ϵ series is h, less than unity, this derivative is always positive. Thus the right member is an increasing analytic function of x. Similarly the left member is an increasing analytic function of y. Thus the above equation determines y as an increasing analytic function of x, and accordingly maps the unit interval on itself in a one to one manner.

To find the transform of a point of the set a_i by this function, we note that since the a_i are enumerable, each a_i is reached at some stage of the approximating process. Thus all the terms after a certain one in the right member contain $x-a_i$ as a factor, and hence vanish when $x=a_i$. Also, from our method of procedure, there is a b_j which when substituted for y causes all the terms in the left member after a certain one to vanish, and makes the sum of those which do not vanish equal the right member with x replaced by a_i. Thus, since we already know that the transformation from x to y is one to one, b_j is the transform of a_i. Similar reasoning shows that each b_j has as its transform some a_i.

Having explicitly constructed a function with the desired properties, we may state

THEOREM I. *For any two enumerable linear point sets, each everywhere dense on an open interval, an analytic function can be found which maps the two intervals on one another, and effects a one to one correspondence between the point sets.*

We may remark in passing that if one of the intervals is infinite both ways, the function we have constructed is only analytic at the points of the open intervals in question; if, however, both intervals are semi-infinite or finite (not necessarily both of the same type) our function is analytic

at the end points of the open interval as well (except, of course, for the pole in the semi-infinite case).

III. Approximation to an analytic function

If the intervals of the above theorem are both finite, we may put a further restriction on the analytic mapping function. In fact, we may have it approximate any given analytic function which maps the intervals on one another.

To see this, let us turn to the mapping function we have constructed in the preceding section which maps the unit interval into itself. We notice that it approximates the function with which we started,

$$y = x.$$

For, our final equation may be written in the form

$$F(y)+y = f(x)+x.$$

As both $F(y)$ and $f(x)$ are dominated by the ε series, and hence numerically less than h, we have

$$|y-x| = |f(x)-F(y)| \leqq |f(x)|+|F(y)| \leqq 2h.$$

Since h was entirely at our disposal, we can take it so that the final function approximates the original one to any desired degree.

A similar relation holds for the derivative, since from

$$F'(y)y'+y' = f'(x)+1$$

we have

$$|y'-1| = \left|\frac{f'(x)-F'(y)}{1+F'(y)}\right| \leqq \frac{2h}{1-h},$$

as $f'(x)$ and $F'(y)$ are both numerically less than h from their definition. Thus y' can be made to approximate unity, by a proper choice of h.

Suppose, now, we were given two enumerable point sets, each everywhere dense on some finite, open interval and an analytic function, $g(x)$, which mapped one of the intervals on the other, and had a derivative which was positive in the corresponding closed interval. We could start with the function

$$y_1 = g(x)$$

and build up a series of approximating functions as in Section II which would map one of the point sets on the other. Of course, h would have to be taken less than the minimum value of $g'(x)$ in the interval, to make the approximations monotonic. If we also took $h < \eta/2$, we would find that, for the final function,

$$|y - g(x)| < \eta.$$

This establishes

THEOREM II. *For any two enumerable linear point sets, each everywhere dense on an open interval, and any analytic function which maps one of the corresponding closed intervals on the other, its derivative being positive in this closed interval, an analytic function can be found which maps the two intervals on one another, effects a one to one correspondence between the point sets, and approximates the given function uniformly.*

For the function we have just constructed, we would also have

$$|y' - g'(x)| \leq \frac{h(1+G)}{1-h},$$

where G is an upper bound for $g'(x)$. Thus h can be chosen so as to make the derivative of the new function approximate $g'(x)$. As we constructed our function, only the first derivatives of the series appearing in the final equation are dominated by the ε series, and hence less than h. By replacing the numerical factors in the denominators of the separate terms by factorials, we can arrange that all such derivatives are so dominated. This enables us to write down equations for the higher derivatives of somewhat similar form to that just given for the first one. Then h can be chosen so as to make any given number of derivatives approximate those of the given function (not an indefinite number, since $(1-h)^m$ appears in the denominator of the bound for the mth derivative), which leads to the

COROLLARY. *The function of Theorem* II *may be so chosen that its first m derivatives (m being any number) approximate those of the prescribed analytic function uniformly.*

IV. APPROXIMATION TO A CONTINUOUS FUNCTION

Instead of starting out with an analytic function which maps our two intervals on one another, we may start with one which is merely continuous, and seek an analytic function which approximates this and takes

our two sets into one another. As we have already shown how to approximate to an analytic function of a certain type, we need merely approximate to the continuous function by one of this type. As the analytic function must map the same interval as the continuous function, i. e., have the same initial and final values, and have a positive derivative throughout the interval, we may not apply the Weierstrass theorem directly, but need an extension of it, to which we proceed.

LEMMA. *Any continuous function which maps one interval on another in a one to one manner preserving sense may be approximated uniformly by an analytic function with positive derivative which maps these intervals on one another.*

The initial step in constructing the function required is to approximate the given continuous function, $c(x)$, by a broken line function, $B(x)$. We take it with the same end points, so that

$$B(a) = c(a), \qquad B(b) = c(b),$$

where a and b are the end points of the interval considered; and also so that throughout the interval

$$|B(x) - c(x)| < \eta/4$$

where η is to be a measure of the final approximation. Since $c(x)$ mapped the intervals on one another in a one to one manner, the segments forming $B(x)$ may be so taken (e. g. as the chords of an inscribed polygon) that their slopes are all positive.

We may obtain an approximation with a continuous derivative by replacing the ends of the chords by small circular arcs, tangent to the chords. They may be taken so small that, if $E(x)$ is the new function,

$$|E(x) - B(x)| < \eta/4.$$

$E(x)$ has a continuous derivative in the closed interval which is always positive. It therefore has a positive minimum, s, so that

$$E'(x) > s > 0.$$

Since the function $E'(x)$ is continuous, by the theorem of Weierstrass, it can be approximated uniformly by an analytic function. Let, then, $F(x)$ be an analytic function, such that

$$|F(x)-E'(x)|<\zeta,$$

where

$$\zeta<s/3 \qquad \text{and} \qquad \zeta<\frac{\eta}{4(b-a)}.$$

Finally, we put

$$G(x) = c(a)+\int_a^x F(x)\,dx+\frac{(x-a)}{b-a}\left[c(b)-c(a)-\int_a^b F(x)\,dx\right].$$

The function $G(x)$ is clearly analytic, and from its form agrees with $c(x)$ at the points a and b. Furthermore,

$$G'(x) = F(x)+\frac{1}{b-a}\left[c(b)-c(a)-\int_a^b F(x)\,dx\right].$$

Since

$$|F(x)-E'(x)|<s/3 \qquad \text{and} \qquad E'(x)>s.$$

$$F(x)>2s/3.$$

Also

$$\left|c(b)-c(a)-\int_a^b F(x)\,dx\right| = \left|\int_a^b [E'(x)-F(x)]\,dx\right| < \frac{s(b-a)}{3}.$$

Consequently

$$G'(x)>2s/3-s/3 = s/3>0,$$

so that $G(x)$ has a positive derivative.

Finally, since

$$E(x) = c(a)+\int_a^x E'(x)\,dx, \qquad \text{and} \qquad c(b)-c(a) = \int_a^b E'(x).$$

$$G(x)-E(x) = \int_a^x [F(x)-E'(x)]\,dx + \frac{x-a}{b-a}\int_a^b [E'(x)-F(x)]\,dx$$

and we have

$$|G(x)-E(x)|<(x-a)\zeta+\frac{x-a}{b-a}(b-a)\zeta<\eta/2.$$

This, combined with our earlier inequalities for $E(x)$ and $B(x)$, shows that

$$|G(x)-c(x)|<\eta,$$

and accordingly $G(x)$ may be taken as the function demanded by the lemma.

By combining the lemma with Theorem II, that is, using the lemma to approximate a continuous function by an analytic function, and then using this as the given analytic function of Theorem II, we obtain

THEOREM III. *For any two enumerable linear point sets, each everywhere dense on an open interval, and any continuous function which maps one of the corresponding closed intervals on the other in a one to one manner which preserves sense, an analytic function can be found which maps the two intervals on one another, effects a one to one correspondence between the point sets, and approximates the given function uniformly.*

One case of this theorem deserves to be specially mentioned. That is, the case in which the two enumerable sets of points become all the rational points in the intervals in question. While we have previously kept the initial and final values of the given function unchanged, it is evident that we can always change the given function by an amount as small as we wish, and bring it about that the initial and final values of the function are rational or irrational according as those of the argument are. This enables us to state

THEOREM IV. *Any continuous function, monotonic in an interval (actually increasing or decreasing, not stationary) may be approximated uniformly in this interval by an analytic function which takes on rational values when. and only when, its argument is rational.*

V. EXTENSIONS TO NON-LINEAR POINT SETS

The theorems we have stated thus far relate to sets of points on segments of straight lines. Similar theorems may be formulated for sets of points on analytic arcs, since by the definition of such an arc it may be mapped on a straight line by an analytic function, and this mapping clearly takes an enumerable everywhere dense set of points on the arc into another such set on the straight line.

If we attempt to extend the theorems to sets of points everywhere dense in a two-dimensional region, we meet difficulties. For, in the process of Section II as applied to an interval, we kept the end points fixed, and thus insured at each stage that the transforms of new points by the function then reached were actually in the region where the points were everywhere dense. As we can not hold the boundary points fixed for a two-dimensional region, the process is no longer applicable. That the proposed generalization of the theorem itself, as well as the method of proof, breaks down, can be seen from a very simple example. Consider two sets of points, each enumerable and everywhere dense inside the unit circle. Any transformation which took one of these sets into the other in a one to one and continuous manner would necessarily be continuous on the boundary of the circle when

extended to all the points in and on the circle. If, now, it was analytic, it would necessarily be a linear fractional transformation, as easily follows from the known theorems on conformal mapping. Let the first set be composed of all the rational points in the unit circle, and the second set consist of all the rational points in the circle and one irrational point. There is no analytic transformation which will take the first set into the second. For, by what has been said, it would have to be a linear fractional transformation. Hence it would preserve the value of the anharmonic ratio of four points. But this ratio is rational for all the points of the first set, while for some groups of points in the second, containing the irrational point, it would be irrational.

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AN ALGEBRAIC SOLUTION OF THE EINSTEIN EQUATIONS*

BY

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The Einstein field equations of gravitation in their cosmological form (for the case where matter is not present) may be written

$$(1) \qquad R_{ik} - \lambda g_{ik} = 0$$

or, what is equivalent,

$$(2) \qquad R_{ik} - \frac{1}{4} g_{ik} R = 0.$$

We wish to present here a new particular solution which is algebraic and extremely simple both in analytic and geometric form, namely

$$(3) \qquad ds^2 = x_1^{-2}(dx_1^2 + dx_2^2) + x_3^{-2}(dx_3^2 + dx_4^2).$$

It is in fact the simplest solution beyond the hypersphere (De Sitter's solution)

$$(4) \qquad ds^2 = x_1^{-2}(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2).$$

Our starting point is to assume the quaternary form

$$g_{ik}\, dx_i\, dx_k$$

to be the sum of two binary forms, one in the variables x_1, x_2, the other in the variables x_3, x_4.

We have then

$$(5) \quad ds^2 = E dx_1^2 + 2 F dx_1 dx_2 + G dx_2^2 + E' dx_3^2 + 2 F' dx_3 dx_4 + G' dx_4^2$$

where E, F, G involve only x_1, x_2 and E', F', G' involve only x_3, x_4. By a transformation of variables we may, without loss of generality, assume

* Presented to the Society, October 29, 1921, and to the National Academy of Sciences, April, 1921.

$$F = 0, \qquad E = G = \mu(x_1, x_2),$$
$$F' = 0, \qquad E' = G' = \nu(x_3, x_4),$$

so that

(6) $$ds^2 = \mu(x_1, x_2)(dx_1^2 + dx_2^2) + \nu(x_3, x_4)(dx_3^2 + dx_4^2).$$

The problem is to find the two functions μ and ν so that equations (2) are satisfied. It is convenient to introduce

(6′) $$\alpha = \frac{1}{2}\log\mu, \qquad \beta = \frac{1}{2}\log\nu.$$

The components of the contracted curvature tensor are easily found to be*

$$R_{11} = R_{22} = \alpha_{11} + \alpha_{22},$$
$$R_{33} = R_{44} = \beta_{33} + \beta_{44},$$

where subscripts applied to α and β denote partial derivatives $\alpha_{11} = \partial^2\alpha/\partial x_1^2$, etc. The other components R_{12}, etc., vanish identically.

The scalar curvature is

$$R = 2\frac{\alpha_{11} + \alpha_{22}}{\mu} + 2\frac{\beta_{33} + \beta_{44}}{\nu}.$$

Substituting in (2) we find merely one condition, which may be written

$$\frac{\alpha_{11} + \alpha_{22}}{\mu} = \frac{\beta_{33} + \beta_{44}}{\nu}.$$

We observe that the left member is a function of x_1, x_2 and the right member is a function of x_3, x_4, therefore both members are equal to a constant, that is, using (6′),

$$e^{-2\alpha}(\alpha_{11} + \alpha_{22}) = c, \qquad e^{-2\beta}(\beta_{33} + \beta_{44}) = c.$$

This means that the curvature of each of the two binary forms (surfaces) is constant. It follows that each may be assumed as the first fundamental quadratic form of a sphere (radius $= c^{-1/2}$). It is, therefore, unnecessary

* We may conveniently apply the formulas given on p. 229 of the author's paper *The solar gravitational field completely determined by its light rays*, Mathematische Annalen, vol. 85 (1922), pp. 227–236.

actually to write out the general solutions of the above partial differential equations. It is sufficient to assume the particular solution

$$2\alpha = \log \frac{1}{cx_1^2}, \qquad 2\beta = \log \frac{1}{cx_3^2}.$$

Our quaternary form (6) is thus

$$ds^2 = \frac{dx_1^2 + dx_2^2}{cx_1^2} + \frac{dx_3^2 + dx_4^2}{cx_3^2}. \tag{7}$$

If the constant c is zero we may take $\alpha = 0$, $\beta = 0$, that is, $\mu = 1$, $\nu = 1$, so that

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

which is euclidean 4-space. Excluding this trivial case every solution of our problem is equivalent to (3) under a homothetic transformation.

If the sum of two independent binary forms is to satisfy (2) *then the forms represent equal spheres and the result is reducible to* (3).

Using another familiar element of a sphere we may write the form (which is equivalent to (7) with $c = 4a$)

$$ds^2 = \frac{dx_1^2 + dx_2^2}{[1 + a(x_1^2 + x_2^2)]^2} + \frac{dx_3^2 + dx_4^2}{[1 + a(x_3^2 + x_4^2)]^2}. \tag{8}$$

Similarly the four-dimensional hypersphere (4) may also be written in the form

$$ds^2 = \frac{dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2}{[1 + a(x_1^2 + x_2^2 + x_3^2 + x_4^2)]^2}. \tag{8'}$$

Thus (8) *is an exact solution of the field equations* (2) *as well as* (8′).

We may interpret our result geometrically in space of six dimensions as follows.

Take a flat space with six cartesian coördinates $X_1, X_2, X_3, X_4, X_5, X_6$. In the 3-flat X_1, X_2, X_3, take a unit sphere

$$X_1^2 + X_2^2 + X_3^2 = 1, \tag{9'}$$

and in the 3-flat X_4, X_5, X_6, another unit sphere

$$X_4^2 + X_5^2 + X_6^2 = 1. \tag{9''}$$

On these spheres as bases construct hypercylinders of five dimensions. The equations of these cylinders are (9′) and (9″). The intersection of these cylinders is the four-dimensional manifold defined by the simultaneous equations

$$X_1^2+X_2^2+X_3^2=1,\qquad X_4^2+X_5^2+X_6^2=1. \tag{10}$$

This is an algebraic four-dimensional manifold of fourth degree which obeys the field equations (2).

Any four-dimensional manifold in a 6-flat which is the intersection of two cylindrical 5-spreads,

$$F(X_1,X_2,X_3)=0,\qquad G(X_4,X_5,X_6)=0, \tag{11}$$

will have for its ds^2 a quaternary form which can be written as the sum of two independent binary forms. Thus we have proved

The only manifolds of type (11) *in flat space of* 6 *dimensions which obey the field equations are reducible to the quartic variety* (10).

Separable forms

A quadratic differential form in n variables may be called *separable* if it can be reduced by any transformation of the variables to the sum of two forms one involving h variables and the other involving k variables, where $h+k=n$.

This we shall call *separable of type* (h,k). The various types for a given n are not necessarily mutually exclusive. There are possibilities of certain special forms belonging to more than one type. We may of course also have separable forms of type $(h_1,h_2,\cdots,h_r)$, where $h_1+h_2+\cdots+h_r=n$. If the form is euclidean it is of type $(1,1,\cdots,1)$, and vice versa this type is obviously euclidean. This is the extreme case of separability, the n-ary form being then transformable into the sum of n independent unary forms.

If $n=4$, the possible types are

(a) $(2,2)$,
(b) $(3,1)$,
(c) $(2,1,1)$,
(d) $(1,1,1,1)$.

When can an Einstein manifold be of one of these types? For (a) the result has been given above. For (b) no solutions exist; the proof is easy but is here omitted.

For (c) no solutions exist except of course those which are of the trivial euclidean type (d). Hence we have the theorem

If an Einstein manifold is to be separable it must be either euclidean or equivalent to (3), *that is the quartic manifold* (10).

The separability so far defined has been *complete.* There is another more general theory of *incomplete or partial separability.* Thus a form in four variables x_1, x_2, x_3, x_4 may in certain cases, even though not separable in the first sense, be reducible for example to the sum of three binary forms

$$ds^2 = Q' + Q'' + Q''', \tag{12}$$

where Q' involves x_1, x_2, Q'' involves x_1, x_3, and Q''' involves x_1, x_4. This refers of course to the coefficients as well as the differentials, that is

$$Q' = g_{11}\, dx_1^2 + 2 g_{12}\, dx_1\, dx_2 + g_{22}\, dx_2^2,$$

where the g's are functions of only x_1, x_2. Einstein solutions of type (12) actually exist. In particular I have found all solutions of the form*

$$ds^2 = \alpha(x_1)\, dx_1^2 + \beta(x_1)\, dx_2^2 + \gamma(x_1)\, dx_3^2 + \delta(x_1)\, dx_4^2. \tag{13}$$

These are included in the type (12). They may be immersed in a flat space of seven dimensions and defined in finite form by means of three surfaces of rotation having a common axis.

* See Science, vol. 54 (1921), p. 304, and American Journal of Mathematics, vol. 43 (1921), p. 220; also a forthcoming paper in these Transactions.

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ELECTRODYNAMICS IN THE GENERAL RELATIVITY THEORY*

BY

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The restricted relativity theory resulted mathematically in the introduction of pseudo-euclidean four-dimensional space and the welding together of the electric and magnetic force vectors into the electromagnetic tensor.

Einstein's general relativity theory led to the assumption that the four-dimensional space mentioned above is a curved space and the curvature was made to account for the gravitational phenomena.

The Riemann tensor which measures the curvature and the electromagnetic tensor seem thus to play essentially different rôles in physics: the former reflects some properties of the space so that gravitation may be said to have been geometricized,—when the space is given all the gravitational features are determined; on the contrary, it seemed that the electromagnetic tensor is superposed on the space, that it is something external with respect to the space, that after space is given the electromagnetic tensor can be given in different ways. Several attempts were made to geometricize the electromagnetic forces, to find a geometric interpretation for the electromagnetic tensor, to incorporate this tensor into the space in the sense in which the gravitational forces had been incorporated.

It seemed that in order to do this it was necessary to change the geometry; to abandon the Riemann geometry and to adopt a more general space with a more complicated curvature tensor, one part of which would then account for the gravitational properties and the other would in the same way account for the electromagnetic phenomena.

H. Weyl arrived in a most natural way to such a generalization. His theory always will remain a brilliant mathematical feat, but it seems that it did not fulfil the expectations as a physical theory and the same seems to be true with respect to other attempts.

The electromagnetic tensor is, however, not entirely independent of the Riemann tensor in the ordinary general relativity theory; these two tensors are connected by the so called energy relation; it seemed to be desirable to try, without breaking the frame of the Riemann geometry, to study

* Presented to the Society, February 24, 1923, December 27, 1923, March 1, 1924, and May 3, 1924.

mathematically the connection between these two most important tensors of physics. This study forms the object of the present paper.

The result of this study is quite unexpected; it is that, under certain assumptions, the electromagnetic field is entirely determined by the curvature of space-time, so that there is no need of further generalizing the general relativity theory; it was only necessary to develop mathematically the consequences of well known relations in order to see that without any modifications it takes care of the electromagnetic field, as far as "classical electrodynamics" is concerned; whether the phenomena of emission and absorption of radiation and such features of the electron theory as equality of charges can be accounted for by the general relativity theory in its original form remains to be seen, but there are indications which show that they might.

As to the method of the study it seemed to me better to avoid, as far as possible, the introduction of things which have no intrinsic meaning, such as coördinates, the g's, the three-indices symbols, the distinction between co- and contravariant quantities, etc. I believe that the present paper shows the advantages of this point of view which I expound at greater length elsewhere.* I also have not used the so called electromagnetic potential vector, which is, moreover, not fully determined; I believe that its use tends to conceal the fundamental properties of the really important things; if we use that vector, the fact that one of the sets of the Maxwell equations is satisfied seems to be granted beforehand and then the other set is a consequence of the general properties of the space;† but in reality the existence of the electromagnetic field imposes on the space additional conditions.

In writing the paper I endeavored not to recede very far from the notation now in general use; I start with components and also translate the results into the language of components, but I hope that the intrinsic meaning of the formulas remains sufficiently clear.

Part I is devoted to the study of the algebraic relations resulting from the energy relation; the electromagnetic tensor in each point is shown to be partly determined by the curvature tensor at that point, only one scalar remaining arbitrary. In Part II by the consideration of differential properties the indeterminateness is reduced to one constant of integration. In Part III it is shown to be possible to eliminate the remaining arbitrariness by consideration of certain integrals.

* American Journal of Mathematics, April, 1924 and January, 1925.

† Cf. Einstein's paper *Bietet die Feldtheorie Möglichkeiten für die Lösung des Quantenproblems*, Berliner Sitzungsberichte, January 15, 1924, statement at the bottom of p. 362.

The contents of Parts I and II were briefly presented in the Proceedings of the National Academy of Sciences in two notes under the title *Electrodynamics in the general relativity theory* in the April and July numbers, 1924. We shall cite them as "First Note" and "Second Note".

PART I. ALGEBRAIC PROPERTIES

1. The invariable plane of a tensor of the second rank

We shall start with the usual form of the general relativity theory; we shall mostly have to consider two tensors of the second rank, the electromagnetic tensor, which is antisymmetric, and the energy tensor which is symmetric. In this first part we shall consider only the connection which exists between these two tensors at a given point, without taking into account the corresponding tensor fields; our considerations will belong, thus, to the algebra of tensors, not to the analysis of tensor fields. But before we consider the relation between our two tensors we shall have to study some geometric properties which belong to every tensor of the second rank.

We shall consider a tensor of the second rank as defining a transformation, and for that purpose it is convenient to use it in its mixed form, f^i_j; if x^i are the contravariant components of a vector (we could also write dx^i) we form the expressions

$$f^i_\rho \, x^\rho \tag{1.1}$$

(we use throughout this paper Greek letters for umbral indices or dummy suffixes); these can be considered as contravariant components of a new vector; we see thus that a tensor of the second rank gives rise to a transformation of a vector into another vector, or to a linear vector function. In many cases it is much more convenient to refer to this linear vector function rather than to the components which depend upon the system of coördinates we are using; we shall simply write x for the vector with the components x^i and $f(x)$ for the transformed vector with the components (1.1); and we shall speak of the tensor f.

We shall write ϱx for the vector whose components are ϱx^i and we shall call the totality of vectors of the form ϱx with a fixed x and a variable ϱ a *direction*; two vectors belong, therefore, to the same direction if their components are proportional. The totality of vectors of the form $\varrho x + \sigma y$ with x and y fixed vectors and ϱ and σ variable numbers will be called a *plane*.

A direction or a plane is called an invariable direction or an invariable plane, respectively, of a tensor f if vectors belonging to it are transformed

by f again into vectors belonging to it (compare, for a general theory of such regions, S. Pincherle and U. Amaldi, *Operazioni Distributive*). If a vector a belongs to an invariable direction of f we have

$$f(a) = \lambda a,$$

where λ is a number which is called the *characteristic* number of this direction. If a belongs to an invariable plane $\varrho x + \sigma y$, we have

$$a = \varrho x + \sigma y, \qquad f(a) = \varrho' x + \sigma' y;$$

applying f to both sides of the second equality and writing $f^2(a)$ for $f[f(a)]$ we find

$$f^2(a) = \varrho'' x + \sigma'' y,$$

and it follows from the last three equalities that a relation of the form

$$f^2(a) - \alpha f(a) + \beta a = 0 \tag{1.2}$$

must hold for a; inversely, if a does not belong to an invariable direction and a relation of the form (1.2) holds, a belongs to an invariable plane defined by the vectors a and $f(a)$.

It is known that a characteristic number λ of an invariable direction satisfies the characteristic equation

$$|f^i_j - \lambda g^i_j| = 0. \tag{1.3}$$

The tensor f itself satisfies a relation which for the four-dimensional space has the form

$$f^4(a) - \alpha f^3(a) + \beta f^2(a) - \gamma f(a) + \delta = 0, \tag{1.4}$$

$f^3(x)$ standing for $f[f^2(a)]$, etc., and the coefficients α, β, γ, δ being equal to the coefficients of the corresponding characteristic equation*.

We shall have to use the following

THEOREM. *Every linear vector function of a four-dimensional space has at least one invariable plane.*

* A very simple proof of this proposition is given by L. E. Dickson, Journal de Mathématiques, ser. 9, vol. 2 (1923), p. 309, footnote.

The proof depends upon the fact that the left hand side of equation (1.4) can be written in the form $h[k(a)]$ or $k[h(a)]$ with

$$h(a) = f^2(a) - \alpha_1 f(a) + \beta_1 a \quad \text{and} \quad k(a) = f^2(a) - \alpha_2 f(a) + \beta_2 a,$$

where α_1, β_1, α_2, β_2 are real numbers. Suppose now one of the functions h and k, say h, never becomes zero for a non-zero argument; then every value of $h(a)$ makes k zero; if among the values of $h(a)$ there are two which belong to different directions, they certainly give us at least one invariable plane; if they all have the same direction we have, e.g., $h(x) = \varrho a$, $h(y) = \sigma a$ and since h never becomes zero ϱ and σ are different from zero; but then we have $h(\sigma x - \varrho y) = 0$, contrary to our assumption. If $h = k$ our equation (1.4) takes the form $h^2(a) = 0$ for every value of a, and from this follows $h(a) = 0$ for every value a.

2. Some properties of the Minkowski space

The scalar product of two vectors x and y can be expressed through their contravariant components in the form

$$xy = x \cdot y = g_{\rho\sigma} x^\rho y^\sigma. \tag{2.1}$$

If we use geodesic coördinates with

$$g_{11} = -1, \; g_{22} = g_{33} = g_{44} = 1 \quad \text{and} \quad g_{ij} = 0 \qquad (i \neq j) \tag{2.11}$$

this gives

$$xy = -x^1 y^1 + x^2 y^2 + x^3 y^3 + x^4 y^4. \tag{2.2}$$

This shows that we have vectors of three kinds: those with negative square, those with positive square and those of zero length. We shall call the latter zero-vectors and the corresponding directions zero-directions. The elementary geometric properties of such a pseudo-euclidean bundle have been well known since the time of Minkowski. We shall only mention that we have three kinds of planes; those which have two zero-directions, those which have none and those which have one; a plane which contains a vector of negative square has two zero-directions.

Given a system of axes we introduce four vectors i, j, k, l by their components

$$1, 0, 0, 0; \qquad 0, 1, 0, 0; \qquad 0, 0, 1, 0; \qquad 0, 0, 0, 1. \tag{2.3}$$

We have

$$i^2 = -1, \qquad j^2 = k^2 = l^2 = 1; \tag{2.4}$$

all the other products are zero. Vice versa, if we have four mutually perpendicular unit vectors (the square of the length of one will then necessarily be -1 and of each of the three others $+1$) we can introduce their directions as axes. We have the relations

$$(2.5)\quad x = ix^1+jx^2+kx^3+lx^4 = -i(ix)+j(jx)+k(kx)+l(lx).$$

If we are given a plane with no zero-direction we can so choose the axes as to make it the k, l plane; a plane with two zero-directions we can make the i, j plane ($i+j$ and $i-j$ being two zero-vectors); a plane with just one zero-direction we can make the $i+j, k$ plane. The last statement may need a proof. Let us take any axes; on the zero-direction of our plane there will be a vector of the form $i+\alpha j+\beta k+\gamma l$ with $\alpha^2+\beta^2+\gamma^2=1$; we can change our "space axis" so as to make $\alpha j+\beta k+\gamma l$ our new j vector; then our zero-vector already has the form $i+j$; now let p be any unit vector of our plane; the vector of this plane $i+j-2p(pi+pj)$ has a zero square and since there is but one zero-direction we must have $pi+pj=0$; if we introduce now the vector $q=(pi)(i+j)+p$, we see that

$$q^2 = 2(pi)(pi+pj)+1=1,\quad qi=-pi+pi=0,\quad qj=pi+pj=0;$$

we can, therefore, choose q for our k.

Once the axes are chosen the formulas can be made more symmetrical, in many cases, by introducing imaginaries, but for the treatment of planes which have just one zero-direction the imaginaries present some difficulties; we shall therefore abstain from introducing them while we have yet to deal with such planes.

3. The antisymmetric tensor of the second rank

If a tensor of the second rank is given in its covariant form f_{ij} or in its contravariant form f^{ij} the property of antisymmetry is simply expressed respectively by the formulas

$$f_{ij} = -f_{ji},\qquad f^{ij} = -f^{ji}.$$

In vector notations we have

$$f(x)\cdot y = f_{\rho\sigma}x^\rho y^\sigma,\qquad f(y)\cdot x = f_{\sigma\rho}x^\rho y^\sigma$$

so that the property of antisymmetry is expressed by the formula

$$(3.1) \qquad f(x)\cdot y = -f(y)\cdot x.$$

Incidentally, for a symmetric tensor we have

$$(3.11) \qquad f(x)\cdot y = f(y)\cdot x.$$

But we have to use, at least temporarily, the mixed form and in this form the property under consideration has a more complicated expression; if we take geodesic coördinates (2.11) we find for the mixed components of an antisymmetric tensor

$$f_1^2 = f_2^1, \quad f_1^3 = f_3^1, \quad f_1^4 = f_4^1, \quad f_2^3 = -f_3^2, \quad f_2^4 = -f_4^2, \quad f_3^4 = -f_4^3.$$

The coefficients are *symmetric* in the indices if one of the indices is 1, and *antisymmetric* in other cases; they are zero when the two indices coincide.

We shall discuss now the question of invariable planes of an antisymmetric tensor. We know that there exists at least one invariable plane (§ 1). Suppose there exists an invariable plane which has no zero-directions; we can take this plane for the k, l plane (§ 2); then we have $f(k) = \alpha k + \beta l$, $f(l) = \gamma k + \delta l$; that means that in the scheme of coefficients

$$(3.2) \qquad \begin{Vmatrix} 0 & A & B & C \\ A & 0 & D & E \\ B & -D & 0 & F \\ C & -E & -F & 0 \end{Vmatrix}$$

$B = C = D = E = 0$. There only remains

$$\begin{Vmatrix} 0 & A & 0 & 0 \\ A & 0 & 0 & 0 \\ 0 & 0 & 0 & F \\ 0 & 0 & -F & 0 \end{Vmatrix}.$$

On the other hand, if there is an invariable plane with two zero-directions we can take it for the i, j plane; we have then $f(i) = \alpha i + \beta j$, $f(j) = \gamma i + \delta j$, so that $B = C = D = E = 0$ with the same result as before. We have therefore in both cases considered

$$f(i) = A\cdot j, \quad f(j) = A\cdot i, \quad f(k) = -F\cdot l, \quad f(l) = F\cdot k.$$

Using (2.5) we find

$$
\begin{aligned}
(3.3)\qquad f(x) &= -Aj(ix) + Ai(jx) - Fl(kx) + Fk(lx) \\
&= A\{i(jx) - j(ix)\} + F\{k(lx) - l(kx)\}.
\end{aligned}
$$

This is the first canonical form of an antisymmetric linear vector function (the same expression holds also for the euclidean bundle where it is the *only* canonical form). The geometric meaning of a function of the form (3.3) is seen to be the following: a vector of each of the invariable planes (the i, j plane and the k, l plane) is transformed into another vector of the same plane, which is perpendicular to the original vector and whose length is, respectively, A or F times greater; the transformation of a vector which does not belong to one of the invariable planes is given by the transformation of its components in these planes.*

The case remains to be considered when the invariable plane or planes have only one zero-direction. According to § 2 we can take such a plane for the $i+j$, k plane; then we have $f(i+j) = \alpha(i+j) + \beta k$, $f(k) = \gamma(i+j) + \delta k$; confronting this with the scheme (3.2) we find $C = E$, $B = D$, $F = 0$, so that

$$f(i) = Aj + Bk + Cl,\ f(j) = Ai - Bk - Cl,\ f(k) = Bi + Bj,\ f(l) = Ci + Cj.$$

It is easy to see that unless $A = 0$ the vectors $f(i)$ and $f(j)$ determine an invariable plane which has two zero-directions, viz. that of the vector $i+j$ and that of the vector $(Aj + Bk + Cl)(B^2 + C^2 - A^2) + (Ai - Bk - Cl)(B^2 + C^2 + A^2)$; A must, therefore, be zero. If now, without changing i and j we choose the unit vector of the direction $Bk + Cl$ for our new k and denote the length of $Bk + Cl$ by G we have

$$f(i) = Gk, \qquad f(j) = -Gk, \qquad f(k) = Gi + Gj, \qquad f(l) = 0,$$

and

$$(3.31)\quad f(x) = G\{i(kx) - k(ix) + j(kx) - k(jx)\} = G\{n(kx) - k(nx)\},$$

* This interpretation was given in a paper presented to the Society, February 24, 1923; compare also Comptes Rendus, vol. 176, p. 1294. A proof for the euclidean case is given by A. Mochoolsky in the Memoirs of the Research Institute, Odessa, February, 1924. It is interesting to note that Sommerfeld originally defined the six-vector as the set of two perpendicular planar quantities (Ebenenstücke), Annalen der Physik, vol. 32 (1910), p. 753; E. T. Whittaker also comes near to this interpretation in his paper on *The tubes of electromagnetic force,* Proceedings of the Royal Society of Edinburgh, vol. 42 (1922), pp. 1-23. See also S. R. Milner's paper in the Philosophical Magazine, ser. 6, vol. 44 (1922), p. 705.

where $n = i + j$; this is the second canonical form of an antisymmetric linear vector function in a pseudo-euclidean bundle. Here we also have two perpendicular planes: $i + j, k$ and $i + j, l$ which are invariable, but this is a different kind of perpendicularity (in both cases we have a so called absolute perpendicularity, i. e., each vector of each of the two planes is perpendicular to each vector of the other plane, but in the first case the two planes have only a common point and in the second they have a common direction).

For the components we have in the first case

$$f_2^1 = f_1^2 = A, \qquad f_4^3 = -f_3^4 = F, \qquad \text{all the others zero;}$$

in the second case

$$f_3^1 = f_1^3 = -f_3^2 = f_2^3 = G. \qquad \text{all the others zero.}$$

Every antisymmetric tensor is known to have two invariants

$$\begin{aligned} I_1 &= f_1^2 f_2^1 + f_1^3 f_3^1 + f_1^4 f_4^1 + f_2^3 f_3^2 + f_2^4 f_4^2 + f_3^4 f_4^3 \qquad \text{and} \\ I_2 &= f_1^2 f_4^3 + f_1^3 f_2^4 + f_1^4 f_2^3 . \end{aligned} \tag{3.4}$$

In the first case their values are $A^2 - F^2$ and AF; in the second case both invariants vanish.

We conceive of an electromagnetic field as of something of the nature of an analytic function (compare Part III); it is natural to assume, therefore, that the invariants of an electromagnetic field cannot be *strictly* zero in one region without being zero all over; and since there are regions where they are different from zero we shall assume that they are different from zero everywhere with the exception only of points. From this point of view a field for which both invariants are strictly zero (a self-conjugate field, using the terminology of H. Bateman*) does not exist in nature and must be considered only as an approximation, this approximation not being, incidentally, an intrinsic quality because it depends on the separation of space and time.

Instead of considering the vanishing of the two invariants I_1 and I_2 as characteristic for the self-conjugate field, we may consider as such the vanishing of one number

$$4\omega^4 = I_1^2 + 4 I_2^2 : \tag{3.5}$$

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* *Electrical and Optical Wave-Motion*, p. 5.

in the case when ω does not vanish, i. e., in the case when the tensor may be presented in the first canonical form, we have

$$\omega^2 = \frac{A^2+F^2}{2}. \tag{3.51}$$

The number $\omega\sqrt{2}$ is considered by Milner (paper cited above) who designates it R.

It is often inconvenient, as already mentioned, to have the square of one of our unit vectors negative while the others have positive squares. In order to avoid this we shall consider henceforth instead of the vector i this vector multiplied by $\sqrt{-1}$, but we shall designate this new vector by the same letter i; this change necessitates the substitution of $-\sqrt{-1}\cdot A$ for A in the formula (3.3); we shall call this imaginary number λ and instead of F we shall write μ. The electromagnetic tensors with which we shall have to deal will, therefore, have the form

$$f(x) = \lambda\{i(jx)-j(ix)\}+\mu\{k(lx)-l(kx)\}, \tag{3.6}$$

with

$$i^2 = j^2 = k^2 = l^2 = 1, \qquad i\cdot j = i\cdot k = \cdots = 0; \tag{3.7}$$

λ is an imaginary and μ a real number.

We shall say that the planes i, j and k, l form the *skeleton* of the tensor; in order to know the tensor it is necessary to know the skeleton and the two numbers λ and μ.

It must be noticed that, whereas λ and μ are entirely determined by the tensor, the vectors i, j, k, l are not; the vectors k, l may be turned in their plane through an arbitrary angle ψ, i. e. we may introduce in their stead two vectors K and L connected with them by the relations

$$k = K\cos\psi - L\sin\psi, \qquad l = K\sin\psi + L\cos\psi; \tag{3.8}$$

the substitution of these expressions in (3.6) will show that the vectors K, L play exactly the same part as k, l. The same can be said with reference to the couple i, j with a little modification necessitated by the fact that i is imaginary; we shall have here the transformation

$$i = I\cos\chi + J\sin\chi\cdot\sqrt{-1}, \qquad j = I\sin\chi\cdot\sqrt{-1}+J\cos\chi. \tag{3.9}$$

I would not say that the consideration of these vectors i, j and k, l instead of the planes which they determine is entirely satisfactory from the point of view of mathematical elegance; it introduces elements which have no intrinsic significance and it is to be hoped that it will eventually be possible to do without them, to operate directly on the planes. But as things stand now, we have to use the vectors. If the two vectors i, j are given they determine also the plane k, l (and vice versa) because there is only one plane perpendicular to a given plane in a four-dimensional bundle. We could, therefore, use only one couple of vectors but this would necessitate the introduction of a new operation and would make our formulas less symmetric.

4. The energy relation

The electromagnetic energy (and momentum) tensor is usually given in the form $f^i_\rho f^\rho_j - \frac{1}{4} g^i_j f^\sigma_\rho f^\rho_\sigma$, but a simpler form* can be obtained if we use the dual or reciprocal tensor d together with f, viz.,

$$\tag{4.1} \frac{1}{2}\{f^i_\rho f^\rho_j - d^i_\rho d^\rho_j\}.$$

Now $f^i_\rho f^\rho_\sigma x^\sigma$ is in vector notation simply $f^2(x)$ because it is the result of the transformation f applied twice; we can write therefore for the energy tensor

$$\tag{4.11} \frac{1}{2}\{f^2(x) - d^2(x)\}.$$

The dual tensor of an antisymmetric tensor f^i_j is defined by

$$d^1_2 = f^3_4,\quad d^1_3 = f^4_2,\quad d^1_4 = f^2_3,\quad d^3_4 = f^1_2,\quad d^4_2 = f^1_3,\quad d^2_3 = f^1_4.$$

If we take f in the canonical form (3.6) its components are

$$\tag{4.2} f^1_2 = \lambda,\qquad f^3_4 = \mu \qquad \text{and} \qquad f^1_3 = f^1_4 = f^4_2 = f^2_3 = 0.$$

The components of d will, therefore, be

$$\tag{4.21} d^1_2 = \mu,\quad d^3_4 = \lambda,\qquad d^1_3 = d^1_4 = d^4_2 = d^2_3 = 0,$$

* See, e. g., J. Rice, *Relativity*, London, 1923, p. 224. This form is due to Laue; see Sommerfeld, loc. cit., p. 768.

so that

$$d(x) = \mu\{i(jx)-j(ix)\}+\lambda\{k(lx)-l(kx)\}. \tag{4.3}$$

It should be noted that with our convention this is an imaginary tensor, i. e., it gives us vectors multiplied by $\sqrt{-1}$; in the formula (4.11) nothing imaginary remains because d is there applied twice in succession. We could of course easily introduce instead of d a real tensor, but we see no harm in its remaining imaginary and in some cases it is even of some advantage (see §§ 6 and 9).

We have

$$f^2(x) = -\lambda^2\{i(ix)+j(jx)\}-\mu^2\{k(lx)+l(kx)\},$$
$$d^2(x) = -\mu^2\{i(ix)+j(jx)\}-\lambda^2\{k(kx)+l(lx)\},$$

so that the expression for the electromagnetic energy tensor (4.11) becomes

$$\omega^2\{i(ix)+j(jx)-k(kx)-l(lx)\}, \tag{4.4}$$

if we put, in accord with (3.51),

$$\omega^2 = \frac{\mu^2-\lambda^2}{2}. \tag{4.5}$$

Now in the general relativity theory the energy tensor at a given point can be calculated from the Riemann tensor. If R^i_j is the contracted Riemann tensor, then the energy tensor is usually assumed to be $R^i_j - \frac{1}{2} g^i_j R^\rho_\rho$. In a region which is free from matter the whole energy is electromagnetic, so that this expression must be equal to the electromagnetic energy tensor and we have the equation

$$R^i_j - \frac{1}{2} g^i_j R^\rho_\rho = f^i_\sigma f^\sigma_j - \frac{1}{4} g^i_j f^\tau_\sigma f^\sigma_\tau. \tag{4.6}$$

Contracting, we see that R^ρ_ρ must be in this case equal to zero, so that the electromagnetic energy tensor is equal to the contracted Riemann tensor. It is also possible to suppose that R^ρ_ρ is a constant different from zero—this would correspond to the cosmological equations. In this case we have to take for the energy tensor the expression $R^i_j - \frac{1}{4} g^i_j R^\rho_\rho$; in both cases, we see, the electromagnetic energy tensor is equal to an expression which can be obtained from the Riemann tensor, i. e., which can be found

if the space-time is given. If we denote this tensor, which is obtained from the curvature of the space-time, by F^i_j we have, therefore, the relation

$$(4.12)\qquad F^i_j = \frac{1}{2}\{f^i_\rho f^\rho_j - d^i_\rho d^\rho_j\} \quad \text{or} \quad F(x) = \frac{1}{2}\{f^2(x) - d^2(x)\}.$$

This relation which we call the energy relation connects the curvature field and the electromagnetic field. We are going to find out what information concerning each of these fields can be obtained from this relation. We shall start by investigating what restrictions are imposed on $F(x)$ by the existence of the relation (4.12). From the general theory of curved space we only know that $F(x)$ is a symmetric linear vector function and that any symmetric linear vector function can be taken for F, as far as general properties of space are concerned; but if we write our relation in the form

$$(4.41)\qquad F(x) = \omega^2\{i(ix) + j(jx) - k(kx) - l(lx)\},$$

we see that $F(x)$ must be a linear vector function of a special form. We are going to ask ourselves how, given a tensor of the second rank, we can know whether it has the form (4.41) or not. First or all, substituting in (4.41) in turn $x = i, j, k, l$, we find

$$(4.42)\qquad F(i) = \omega^2 i, \quad F(j) = \omega^2 j, \quad F(k) = -\omega^2 k, \quad F(l) = -\omega^2 l.$$

We see that the vectors i, j, k, l, belong to invariable directions, their characteristic numbers being $\omega^2, \omega^2, -\omega^2, -\omega^2$. It is easy to see that every direction of each of the planes i, j and k, l is an invariable direction with the characteristic number $\omega^2, -\omega^2$, respectively. Here we have a full geometric characterization of F:

(4.43) *It has two planes of invariable directions with characteristic numbers of opposite signs; these planes are (absolutely) perpendicular with one common point.*

If we want to find a characterization of F in terms of its components, the best way is to start with the remark that

$$(4.7)\qquad \begin{aligned} F^2(x) = F[F(x)] &= \omega^2\{F(i)(ix) + F(j)(jx) - F(k)(kx) - F(l)(lx)\} \\ &= \omega^4\{i(ix) + j(jx) + k(kx) + l(lx)\} = \omega^4 x. \end{aligned}$$

In components we may write for the left hand part (as we did before for f) $F^i_\rho F^\rho_\sigma x^\sigma$, and the right hand part may be written as $\omega^4 g^i_\rho x^\rho$; we therefore have

$$(4.71)\qquad F^i_\rho F^\rho_j = g^i_j \omega^4.$$

This is a necessary condition for F but it is not sufficient, as, e. g., the function

$$F(x) = \omega^2\{i(ix)-j(jx)-k(kx)-l(lx)\}$$

also satisfies it. But together with the condition

$$F_\rho^\rho = 0, \tag{4.8}$$

which was obtained by contracting (4.6), the equation (4.71) gives a full characterization of F. The proof of this statement will be a little cumbersome because we have not studied the geometric properties of a symmetric function in a pseudo-euclidean bundle. We know, however, that $F(x)$, like every linear vector function, has at least one invariable plane; if there is such a plane with two zero-directions we take it for the i, j plane; we have $F(i) = \alpha i + \beta j$, $F(j) = \gamma i + \delta j$. Comparing this with the scheme for a symmetric linear vector function, viz.,

$$F(i) = Ai + Bj + Ck + Dl, \qquad F(j) = Bi + Ej + Gk + Hl,$$
$$F(k) = Ci + Gj + Kk + Ll, \qquad F(l) = Di + Hj + Lk + Ml,$$

we find $C = D = G = H = 0$; we thus have two perpendicular invariable planes and we shall show that each of them has two perpendicular invariable directions with the characteristic numbers $\pm\omega^2$. Take, e. g., the plane i, j; writing that $F^2(i) = \omega^4 i$, $F^2(j) = \omega^4 j$, we find $A^2+B^2 = B^2+E^2 = \omega^4$, $B(A+E) = 0$; if $B = 0$, $A^2 = E^2 = \omega^4$ and the vectors i and j give us the directions we want. If $B \neq 0$, $E = -A$ and a simple calculation shows that the vectors $Bi-(A-\omega^2)j$ and $(A-\omega^2)i+Bj$ belong to two invariable directions with the characteristic numbers $\pm\omega^2$. We have thus established that there are four mutually perpendicular directions with characteristic numbers $\pm\omega^2$. The equation (4.8) shows that the sum of the characteristic numbers is zero; two of them must therefore be positive and two negative. It remains to show that there always is a plane with two zero-directions; if all time-directions are invariable a plane defined by two of them is certainly invariable and it contains two zero-directions (§ 2); if there is a time-direction which is not invariable let the vector i belong to it; the plane determined by i and $F(i)$ is invariable because $F[F(i)] = F^2 i = \omega^4 i$, and it has two zero-directions.

The above discussion leaves open the possibility $F(i) = -\omega^2 i$; if we want to exclude this we have to put down the additional condition

$$F_1^1 > 0. \tag{4.9}$$

We know now the necessary and sufficient conditions which have to be satisfied if our tensor F is to have the form (4.41); if these conditions are satisfied we can find the vectors i, j, k, l and the number ω. Once we have found them we put, to satisfy (4.5),

$$\lambda = \omega\sqrt{-2}\sin\varphi, \qquad \mu = \omega\sqrt{2}\cos\varphi, \tag{4.51}$$

where φ is an arbitrary (real) angle and have in

$$f(x) = \lambda\{i(jx) - j(ix)\} + \mu\{k(lx) - l(kx)\}$$

an electromagnetic tensor which satisfies the energy relation with the given tensor F. We see thus that the electromagnetic tensor is not entirely determined by the curvature tensor of space-time at the same point; after the curvature tensor is given there are an infinity of electromagnetic tensors which are possible from the point of view of the energy relation. To complete the determination of the electromagnetic tensor we must know, besides the curvature tensor, the number φ. From the geometric point of view we may say that the curvature tensor gives the skeleton of the electromagnetic tensor, but instead of giving the two numbers λ and μ it gives only their combination $\mu^2 - \lambda^2$.

It would, however, be wrong to conclude from this that the curvature of space-time does not determine the electromagnetic *field*. So far we have considered only the relation between the two tensors *in a point*. We shall now take into account their differential properties.

PART II. DIFFERENTIAL PROPERTIES

5. Preliminary remarks

We shall proceed to study a region of space-time, in each point of which we consider the electromagnetic tensor; in each point the energy relation holds, so that the results of Part I are applicable, but we shall now take into account also the Maxwell equations which are satisfied by the electromagnetic tensor. We shall ask ourselves, first, what additional information with respect to the field F can be obtained from the fact that f, which is connected with F by the energy relation, is, at the same time, subjected to the Maxwell* equations. After we have found the restrictions which have to be imposed on the field of F we shall, secondly, take up again the question of how far the field f is determined by the field F; and finally

* When we say Maxwell equation in the following we always imply *in empty space*.

we shall translate the conditions for the field F into the language of components.

The usual form of the Maxwell equations in regions where matter is absent is

$$f^{\rho}_{i,\rho} = 0, \qquad d^{\rho}_{i,\rho} = 0, \tag{5.1}$$

where d, as before, means the dual tensor of f, and the index after the comma corresponds to covariant differentiation. It will not, however, be convenient for us to deal with the components of tensors; the results of § 3 permit us, it is true, to choose the coördinates for a given point in such a way as to bring the components of f into the simple form (see 4.2)

$$\begin{Vmatrix} 0 & \lambda & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu \\ 0 & 0 & -\mu & 0 \end{Vmatrix}, \tag{5.2}$$

but we shall not be able to use this form where differentiation is involved, because this holds only for the point in which the system of coördinates is geodesic and if it is geodesic for one point it cannot be geodesic in its neighborhood. We therefore take the form (see (3.6) and (4.3))

$$\begin{aligned} f(x) &= \lambda\{i(jx)-j(ix)\}+\mu\{k(lx)-l(kx)\}, \\ d(x) &= \mu\{i(jx)-j(ix)\}+\lambda\{k(lx)-l(kx)\}. \end{aligned} \tag{5.3}$$

We can consider f and d as given in this form for all points (of a certain region). Of course, the vectors i, j, k, l will not be the same in different points; in a curved space there is no such thing as equality—and still less identity—between vectors of different bundles. The values of the numbers λ and μ may also change from point to point. The vectors i, j, k, l and the numbers λ and μ will therefore be point functions. If a definite system of coördinates is introduced, the numbers λ and μ and the components of the vectors i, j, k, l will be functions of coördinates; they will constitute tensor fields of rank zero and one, respectively. Of course the tensor analysis can be developed from the beginning independently of coördinates (compare the author's papers cited in the introduction), but here we shall translate into vector language only the things which we are going to use.

If we have a tensor of rank zero, i. e., a point function λ, it has in each point four derivatives λ_i; we may consider them as the covariant components of a vector, which is called the gradient of λ and denoted by grad λ. We may, on the other hand, consider instead of the derivatives the differential; if we denote the differentials of coördinates by h^i and the vector which has h^i for its components by h, the differential can be written as $\lambda_\rho h^\rho$. This is the scalar product of grad λ by h; it is also a scalar linear function of h which we shall designate $\lambda'(h)$: in short, the differential of a scalar field λ is

$$\lambda'(h) = \operatorname{grad} \lambda \cdot h = \lambda_\rho h^\rho; \tag{5.4}$$

and we have, using (2.5) (with — changed into + according to (3.7)),

$$\operatorname{grad} \lambda = i \cdot \lambda'(i) + j \cdot \lambda'(j) + k \cdot \lambda'(k) + l \cdot \lambda'(l), \tag{5.5}$$

if i, j, k, l are any four perpendicular unit vectors.

If we have a vector field v, i. e., a tensor field of rank one with contravariant components v^i, the absolute derivatives $v^i_{;j}$ of these components can be considered as mixed components of a tensor of the second rank; if instead of the derivatives we consider the differential $v^i_{;\rho} h^\rho$ we can interpret this as a transformation applied to the vector h, i. e., a linear vector function, which we shall designate by $v'(h)$.

If we have a tensor field of the second rank given by its mixed components f^i_j the absolute derivatives will be $f^i_{j,k}$ and if we consider instead of the components f^i_j the transformation $f^i_\rho x^\rho$, the differential will be $f^i_{\rho,\sigma} x^\rho h^\sigma$, i. e., a bilinear vector function with the arguments x and h; we shall in vector notations write for the differential of the linear vector function $f(x)$ simply $f'(x, h)$.

The result of contracting $f^i_{j,k}$ with respect to the indices i, k is $f^\rho_{j,\rho} = f^1_{j,1} + f^2_{j,2} + f^3_{j,3} + f^4_{j,4}$; these are components of a vector, say v_j. It is easy to see that in vector notations this becomes

$$v = f'(i, i) + f'(j, j) + f'(k, k) + f'(l, l). \tag{5.6}$$

We shall not go farther in this direction; that is all we need for the translation of the Maxwell equations. But before we start the work on them we notice that, differentiating the identities (3.7), we find

$$i'(h)\cdot i = j'(h)\cdot j = k'(h)\cdot k = l'(h)\cdot l = 0; \tag{5.7}$$

$$\begin{aligned} i'(h)\cdot k + k'(h)\cdot i = j'(h)\cdot k + k'(h)\cdot j &= i'(h)\cdot l + l'(h)\cdot i \\ &= j'(h)\cdot l + l'(h)\cdot j = 0; \end{aligned} \tag{5.8}$$

$$i'(h)\cdot j + j'(h)\cdot i = k'(h)\cdot l + l'(h)\cdot k = 0. \tag{5.9}$$

6. Geometric properties of a Maxwell field

In order to write down the first set of Maxwell's equations in vector form we have to write that the vector v (5.6) is zero when f is the tensor given by (5.3). The differential of $f(x)$ is

$$\begin{aligned} f'(x, h) = {} & \lambda'(h)\cdot i\cdot(jx) - \lambda'(h)\cdot j\cdot(ix) + \mu'(h)\cdot k\cdot(lx) - \mu'(h)\cdot l\cdot(kx) \\ & + \lambda\cdot i'(h)\cdot(jx) - \lambda\cdot j'(h)\cdot(ix) + \mu\cdot k'(h)\cdot(lx) - \mu\cdot l'(h)\cdot(kx) \\ & + \lambda\cdot i\cdot[j'(h)\cdot x] - \lambda\cdot j\cdot[i'(h)\cdot x] + \mu k[l'(h)\cdot x] - \mu l[k'(h)\cdot x]. \end{aligned} \tag{6.1}$$

Instead of writing that the vector v is zero we shall write that its components, i. e., the products $v\cdot i$, etc., are zero. In order to form, e. g., $v\cdot i$, we consider $f'(x, h)\cdot i$; the multiplication of (6.1) by i destroys on its left hand side all the terms which are perpendicular to i, i. e., those which have the directions of j, k, l, i'. There remains

$$\begin{aligned} f'(x, h)\cdot i = {} & \lambda'(h)\cdot(jx) - \lambda\cdot[j'(h)\cdot i]\cdot(ix) + \mu\cdot[k'(h)\cdot i]\cdot(lx) \\ & - \mu\cdot[l'(h)\cdot i]\cdot(kx) + \lambda[j'(h)\cdot x]. \end{aligned} \tag{6.2}$$

To obtain $v\cdot i$ we have to put here $x = h$, to substitute for this vector in turn i, j, k, l and to add the results. The first term of (6.2) gives a vector different from zero only for $h = j$; the second for $h = i$, the third for $h = l$, the fourth for $h = k$ and the last for $h = i$, or k or l. We have thus, since the second and the last terms of the result destroy each other, using (5.8)

$$v\cdot i = \lambda'(j) - \lambda\{k'(k)\cdot j + l'(l)\cdot j\} - \mu\{l'(k)\cdot i - k'(l)\cdot i\} = 0. \tag{6.3}$$

This is a scalar equation and we shall have three more similar equations for the other components of v from the first set of the Maxwell equations (5.1) and four more from the second set. We can obtain them from (6.3) by interchanging i and j, k and l, and λ and μ. But we need now only the one which we obtain by interchanging λ and μ, viz.

$$\mu'(j) - \mu\{k'(k)\cdot j + l'(l)\cdot j\} - \lambda\{l'(k)\cdot i - k'(l)\cdot i\} = 0. \tag{6.31}$$

Eliminating from (6.3) and (6.31) first the third terms and then the second terms, we obtain

$$\mu\mu'(j) - \lambda\lambda'(j) = (\mu^2 - \lambda^2)\{k'(k)\cdot j + l'(l)\cdot j\}, \tag{6.41}$$

$$\lambda\mu'(j) - \mu\lambda'(j) = (\lambda^2 - \mu^2)\{k'(l)\cdot i - l'(k)\cdot i\}. \tag{6.42}$$

Now we have from (4.5) and (4.51), remembering that $\omega \neq 0$,

$$\frac{\mu\mu' - \lambda\lambda'}{\mu^2 - \lambda^2} = \frac{1}{2}\,\frac{(\mu^2 - \lambda^2)'}{\mu^2 - \lambda^2} = \frac{1}{2}\,\frac{(\omega^2)'}{\omega^2} = \frac{\omega'}{\omega},$$

$$\frac{\lambda\mu' - \mu\lambda'}{\mu^2 - \lambda^2} = \frac{1}{2\omega^2}\begin{vmatrix} -\omega\sqrt{2}\sin\varphi\cdot\varphi' + \omega'\sqrt{2}\cos\varphi & \omega\sqrt{-2}\cos\varphi\cdot\varphi' + \omega'\sqrt{-2}\sin\varphi \\ \omega\sqrt{2}\cos\varphi & \omega\sqrt{-2}\sin\varphi \end{vmatrix}$$

$$= -\varphi'\sqrt{-1}.$$

This permits us to write (6.41) and (6.42) in the form

$$\frac{\omega'(j)}{\omega} = k'(k)\cdot j + l'(l)\cdot j = [k'(k) + l'(l)]\cdot j,$$

$$\varphi'(j)\cdot\sqrt{-1} = k'(l)\cdot i - l'(k)\cdot i = [k'(l) - l'(k)]\cdot i,$$

and we have three more of each type. If we put

$$\begin{aligned} p = {} & i\cdot[k'(k)\cdot i + l'(l)\cdot i] + j\cdot[k'(k)\cdot j + l'(l)\cdot j] \\ & + k\cdot[i'(i)\cdot k + j'(j)\cdot k] + l\cdot[i'(i)\cdot l + j'(j)\cdot l], \end{aligned} \tag{6.51}$$

$$\begin{aligned} q = {} & i\cdot[k'(l)\cdot j - l'(k)\cdot j] + j\cdot[l'(k)\cdot i - k'(l)\cdot i] \\ & + k\cdot[i'(j)\cdot l - j'(i)\cdot l] + l\cdot[j'(i)\cdot k - i'(j)\cdot k], \end{aligned} \tag{6.52}$$

and use (5.5) we find *as the equivalents of Maxwell's equations*

$$\operatorname{grad}\omega = \omega p \qquad \text{or} \qquad \operatorname{grad}\log\omega = p, \tag{6.61}$$

$$\sqrt{-1}\operatorname{grad}\varphi = q. \tag{6.62}$$

In § 3, we called the two invariable planes of an antisymmetric tensor the skeleton of this tensor. We shall now call skeleton of an antisymmetric

field the totality of the skeletons of its tensors. It is easy to show that the vectors p and q defined by (6.51) and (6.52) are entirely determined (for each point) by the skeleton of the field (in the neighborhood of that point); in order to do so it is enough to notice that the form of the expressions (6.51), (6.52) is not changed by the transformations (3.8) and (3.9). The equations (6.61) and (6.62) give therefore a property of the skeleton of an antisymmetric field which satisfies Maxwell's equations, which may be stated as follows:

THEOREM. *If an antisymmetric field satisfies Maxwell's equations, the vectors p and q defined by its skeleton are gradients of scalar functions.*

The converse is also true. Suppose we are given two perpendicular planes in each point of a region and we want to know whether there exists a Maxwellian field which has these planes for its skeleton. We choose in each plane two perpendicular unit vectors i, j and k, l respectively, and form according to the formulas (6.51) and (6.52) the vectors p and q; if these vectors are gradients of scalar functions there exists an ∞^2 of different Maxwellian fields with these planes as skeletons. In fact, we can determine two functions ω and φ (each containing an arbitrary additive constant), satisfying (6.61) and (6.62); if we now form λ and μ according to the expressions (4.51) and use them in (5.3) we have the fields in question.

It is interesting to notice that q is an imaginary vector, i. e. a vector of our space multiplied by $\sqrt{-1}$, because i enters in every term once as a factor (p is real because i enters in some of its terms twice and does not enter in other terms at all). If we consider, in a purely formal way, the sum $p+q$ as a complex vector we can say that it is the gradient of

$$\log \omega + \varphi \sqrt{-1} = \log \omega e^{\varphi\sqrt{-1}}.$$

The formulas (4.51) show that

$$\omega e^{\varphi\sqrt{-1}} = \mu + \lambda.$$

Following this line and introducing complex tensors we could considerably simplify our calculations but as the purpose of this paper is only to show how the electromagnetic field is determined by the curvature it does not appear desirable to make the calculations depend on these concepts because this would tend to obscure the principal point at issue. (See, however, § 9.)

A different expression is given for the vector p (with the sign changed) in the "First Note" (formula 4). This expression holds only if the vectors i, j, k, l are chosen in a special way indicated there and does not seem to have any essential advantage over (6.51).

7. Differential properties of the energy tensor

We saw (end of § 4) that the curvature tensor gives the skeleton of the electomagnetic tensor and the number ω^2 in each point. We can restate this now saying that the curvature *field* determines the skeleton of the electromagnetic *field* and the scalar *function* ω^2. In order to complete the determination of the electromagnetic field it remains for us to determine the function φ, but we shall take this question up a little later. For the present we emphasize the fact that the equations (6.61) and (6.62) must furnish us some properties of space-time in which there is an electromagnetic field, because the vectors p, q and the function ω^2 are determined by the curvature of space-time.

As for the equation (6.61) both p and ω are given by the curvature so that it directly gives us a property of space-time. *This property is, however, not new*; it is a consequence of the known relation

$$\left(R_i^\rho - \frac{1}{2} g_i^\rho R_\sigma^\sigma\right)_{i,\varrho} = 0, \tag{7.1}$$

which holds in every curved space.* In our case this relation takes the simpler form

$$F_{i,\rho}^\rho = 0. \tag{7.2}$$

Using the expression (4.41) for F and proceeding in the same way as we did in the beginning of § 6 when we were about to translate Maxwell's equations, which have the same form as (7.2), we find

$$\begin{aligned} F'(x, h)\cdot i &= 2\omega\cdot\omega'(h)\cdot(ix) \\ &+ \omega^2\{[j'(h)\cdot i](jx) - [k'(h)\cdot i](kx) - [l'(h)\cdot i](lx) + i'(h)\cdot x\} \end{aligned} \tag{7.3}$$

and

$$\begin{aligned} v\cdot i &= 2\omega\cdot\omega'(i) \\ &+ \omega^2\{j'(j)\cdot i - k'(k)\cdot i - l'(l)\cdot i + i'(j)\cdot j + i'(k)\cdot k + i'(l)\cdot l\} = 0. \end{aligned}$$

The relations (5.9) show that the first and the fourth terms in the brackets give a sum zero, and the relations (5.8) that the fifth is equal to the second and the sixth to the fourth; we can write therefore, remembering that $\omega \neq 0$,

$$\frac{\omega'(i)}{\omega} = k'(k)\cdot i + l'(l)\cdot i.$$

* Cf. J. A. Schouten and D. J. Struik, Philosophical Magazine, vol. 47 (1924), p. 584.

and, with the three other similar equations, this is equivalent to (6.61); this proves our assertion that this last equation does not impose any new restrictions on space-time. It may, however, be argued that the choice of the expressions for the energy tensor in terms of the curvature tensor, viz. R^i_j or $R^i_j - \frac{1}{2} g^i_j R^\sigma_\sigma$ was influenced by the consideration that for the energy tensor the equation (7.2) must be satisfied.

We shall try now to find whether equation (6.62) gives us some property of space-time containing an electromagnetic field. We know that the point function φ which enters in (6.62) is *not* determined by the tensor F in the corresponding point; we have, therefore, to eliminate φ from this equation and this we can do simply saying that q must be a gradient of a scalar field; or we may write

$$(7.4) \qquad \operatorname{rot} q = 0, \qquad \text{or} \qquad q_{i,j} = q_{j,i}.$$

This property of space-time containing an electromagnetic field does not seem to be a consequence of general properties of curved space; it seems to be an additional restriction imposed on our space-time. However this may be, we suppose henceforth that this condition is satisfied.

We return now to the question of how far the electromagnetic field is determined by space-time. We stated at the beginning of this section that we had still to determine the function φ; *but that is just what equation* (6.62) *does*; it determines the function φ, the only remaining arbitrariness being in a constant of integration. If φ is a solution of (6.62) the general solution is $\varphi + \gamma$, γ being a constant. From (4.51) we obtain

$$(7.5) \qquad \lambda = \omega\sqrt{-2}\sin(\varphi+\gamma), \qquad \mu = \omega\sqrt{2}\cos(\varphi+\gamma),$$

and, if by λ_0 and μ_0 we designate the values of λ, μ which correspond to $\gamma = 0$, we have

$$\lambda = \lambda_0 \cos\gamma + \mu_0 \sin\gamma\sqrt{-1}, \qquad \mu = \mu_0\cos\gamma + \lambda_0 \sin\gamma\sqrt{-1};$$

if, further, by f_0 and d_0 we designate the tensor fields which are obtained from (5.3) for $\lambda = \lambda_0$, $\mu = \mu_0$, we can write *the general electromagnetic field which is compatible with the given space-time in the form*

$$(7.6) \quad f = f_0\cos\gamma + d_0 \sin\gamma\cdot\sqrt{-1}, \qquad d = d_0\cos\gamma + f_0\sin\gamma\cdot\sqrt{-1},$$

the vectors i, j, k, l and the number ω being determined by the tensor F in the point considered and the function φ by the field F in the neighborhood of that point.

It is not the place here to treat the connection of the results obtained with the question of radiation, which was briefly indicated in our "Second Note."

8. Second order property in components

The field F determines the skeleton, the skeleton determines the vector field q; equation (7.4) expresses, therefore, a property of the tensor field F. We shall show now how this property can be expressed in terms of the components of F.

Multiplying both sides of the relation $F^2(x) = \omega^4 x$ (see (4.7)) by y and using the symmetry of F (3.11), we obtain

$$F(x)\cdot F(y) = \omega^4\cdot(xy).$$

In what follows we will consider only the vectors i, j, k, l, which are mutually perpendicular, as the values of x, y; therefore if x, y are different we will have

$$F(x)\cdot F(y) = 0.$$

Differentiating this we obtain

$$F'(x, h)\cdot F(y) + F'(y, h)\cdot F(x) = 0. \tag{8.1}$$

We now form

$$P(x, y, z) = F'(x, y)\cdot F(z) + F'(y, z)\cdot F(x) + F'(z, x)\cdot F(y). \tag{8.2}$$

Using (8.1) we easily see that $P(x, y, z)$ changes its sign when two of its arguments are interchanged (always supposing x, y, z to be three different vectors from among i, j, k, l); it has, therefore, only four essentially different values, but they can be obtained from one by interchanging i, j, k, l. Let us calculate, e. g., $P(i, j, k)$, or, according to (4.42),

$$\omega^2\{-F'(i,j)\cdot k + F'(j, k)\cdot i + F'(k, i)\cdot j\};$$

the middle term can be obtained from (7.3), making $x = j$, $h = k$. Taking in consideration (5.9) we see that it vanishes. To obtain the last term, we interchange in (7.3) i and j, and make then $x = k$, $h = i$; there remains

$$\omega^2\{-k'(i)\cdot j + j'(i)\cdot k\} = 2\omega^2[j'(i)\cdot k]$$

according to (5.8). With the aid of (8.1) we see that $F'(i, j)\cdot k$ can be obtained from this interchanging i and j. We have thus

$$P(i, j, k) = 2\omega^4[j'(i)\cdot k - i'(j)\cdot k].$$

Confronting this with (6.52) we see that this is the product $q \cdot l$ multiplied by the factor $2\omega^4$, or that $P(i, j, k)$ is but for this factor the l-component of the vector q.

If we make $x = i$, $y = j$, $z = k$ in (8.2) we obtain for $P(i, j, k)$ an expression which, translated in the usual language of coördinates, is

$$F^{\rho}_{1,2}\, F_{\rho 3} + F^{\rho}_{2,3}\, F_{\rho 1} + F^{\rho}_{3,1}\, F_{\rho 2}:$$

this is equal to $\omega^4 q_4$ but it is a component of a tensor of the third rank, which, according to our remark following (8.2), is completely alternating. It is, therefore, more convenient to introduce instead of q a completely alternating tensor of the third rank q_{ijk} defined for geodesic coördinates by the equalities

(8.3) $$q_{123} = q_4, \quad q_{234} = -q_1, \quad q_{341} = q_2, \quad q_{412} = -q_3,$$

and which is sometimes referred to as complement of q_i. For this tensor we have then*

(8.4) $$2\omega^4 \cdot q_{ijk} = F^{\rho}_{i,j}\, F_{\rho k} + F^{\rho}_{j,k}\, F_{\rho i} + F^{\rho}_{k,i}\, F_{\rho j}.$$

It remains to write in terms of the tensor q_{ijk} the equations (7.4), which express the condition that q must be a gradient. Take, e. g., the equation $q_{1,2} = q_{2,1}$; using (8.3) we obtain $-q_{234,2} = q_{341,1}$ or, on account of the alternating property, $q_{341,1} + q_{342,2} = 0$; and finally since q_{ijk} vanishes when two indices are equal,

$$q^{341}{}_{,1} + q^{342}{}_{,2} + q^{343}{}_{,3} + q^{344}{}_{,4} = 0,$$

where we use contravariant components, which makes no difference while we are using geodesic coördinates but permits us to write the result in a form which is independent of the system of coördinates, viz.,

(8.5) $$q^{ij\tau}{}_{,\tau} = 0.$$

This together with the formula (8.4) defining q_{ijk} gives us the differential conditions to which the curvature tensor is subjected as a consequence of the presence of the electromagnetic field.

*In the "Second Note", formula (11), ω^4 must stand instead of ω^2; this is obvious because q must not change when F is multiplied by a constant.

PART III. INTEGRAL PROPERTIES AND SINGULARITIES

9. Analogy with analytic functions

In order to find the significance of the fact that the curvature of space-time seems to leave undetermined a constant in the expression for the electromagnetic field we shall have to touch upon the question of *matter*, which we consider as constituted by the singularities of the field. In discussing these singularities much help can be derived from the consideration of both points of striking *analogy* and points of *difference* between the theory of the electromagnetic field and the theory of analytic functions of a complex variable.

We begin with the *analogy*, which can also be stated by saying that both the theory of analytic functions and the theory of the electromagnetic field are special cases, corresponding to $r = 2$, and $r = 4$, respectively, of a general theory of conjugate functions, imagined by Volterra as early as 1889.* From this point of view the Maxwell equations are analogous to the Cauchy-Riemann equations of the theory of functions. They can also be replaced by an equivalent integral relation which is analogous to the Cauchy-Morera theorem of the theory of functions. Before we write down this integral form of the Maxwell equations, we go one step farther than is usually done (so far as we know†) and introduce, instead of the tensors f and d, their sum

$$(9.1) \qquad w(x) = f(x) + d(x) = \nu\{i(jx) - j(ix) + k(lx) - l(kx)\}.$$

Since the tensor d is an imaginary tensor (cf. the statement after (4.3)) the tensor w is to be considered as a *complex* tensor, i. e., if x is a vector of our space, $w(x)$ is the sum of a vector of our space and of a vector of our space multiplied by $\sqrt{-1}$; the number ν, being the sum of a real number μ and an imaginary number λ, is also a complex number. Incidentally, from this point of view the tensor F is the product of w by the conjugate tensor $\bar{w}$, or the square of the modulus of the tensor w, and the number ω^2 is half the square of the modulus of ν; using these notations we could have simplified our calculations in the §§ 6 and 7. grad log ν would furnish us the complex vector $p + q$, etc.

* Lincei Rendiconti, 1889, 1st semester, pp. 599–611 and 630–640. The analogy in question has been already noticed. See, e. g., F. Kottler, *Maxwell'sche Gleichungen und Metrik*, Wiener Sitzungsberichte, IIa, vol. 131, No. 2, pp. 119–146. This paper contains full bibliographical references.

† Compare, however, L. Silberstein, Annalen der Physik, ser. 4, vol. 22 (1907), p. 579 and H. Weber, *Partielle Differentialgleichungen der Mathematischen Physik*, vol. 2, 1901, p. 348.

We can formulate now the analogue of the Cauchy-Morera theorem as follows; the statement that Maxwell's equations for empty space hold in a certain region is equivalent to the statement that the integral

$$\int w_n \, d\sigma, \tag{9.2}$$

taken over any two-dimensional surface which belongs to the region and can be continuously transformed into a point without leaving that region, vanishes.* An immediate consequence of this is that the value of integral (9.2) when it does not vanish—this value is a complex number—does not change if, instead of one closed surface, we take another into which the former can be continuously transformed without leaving the region where Maxwell's equations are satisfied.

The question naturally arises: what is it in this theory, that takes the place of singular points of the theory of analytic functions? Many considerations, both physical and mathematical, lead us to believe that the most interesting objects of this kind are singular lines (having time-direction). If we consider a two-dimensional surface Σ which surrounds such a singular line Γ (much as a circle surrounds a straight line in our three-dimensional space), the value of the integral (9.2) taken over Σ is not necessarily zero, but it follows from what was said after (9.2) that we may change Σ as we want; so long as it surrounds Γ and remains in a simply connected region in which Γ is the only singularity, the value of the integral will not change. In other words this value is entirely determined by the singular line. This value, which is a complex number, is obviously an analogue of the residue, and we shall use for it this word.

Now it so happens that, if we look for the physical interpretation of (9.2), we find that its real part gives the electric charge which is present in some three-dimensional volume enclosed by our surface, and the imaginary part would correspond to a magnetic charge, *but this magnetic charge is always zero.* This last fact seems to be inexplicable from the point of view of the electromagnetic field considered independently of the curvature of space-time, or, let us say, in the space-time of special relativity theory. But it is different from the point of view of general relativity theory on which we stood in the first two parts of the present paper, and to which we shall revert presently.

* A formulation of Maxwell's equations involving integrals over two-dimensional surfaces in time-space was given by R. Hargreaves as early as 1908 (contemporaneously with the famous publications of Minkowski) in the Cambridge Philosophical Society Transactions, vol. 21, p. 116. For a comprehensive presentation see F. D. Murnaghan's book *Vector Analysis and the Theory of Relativity*, Baltimore, 1922, especially p. 72 sqq.

10. Consequences of the curvature of space-time

The considerations regarding integral properties of the electromagnetic field and the residue are independent of the metrical structure of the space. They are, therefore, applicable in the case when the space is the Riemann space of the general relativity theory (in fact, Volterra's general theory holds in much more general spaces). If we consider space-time as originally given, the electromagnetic field is, as we saw in § 7, not completely determined by it; we may say that there is an infinity of possible electromagnetic fields which are given by the expressions (7.6) involving the arbitrary constant γ. All these "associated" fields will have, obviously, the same singular lines but the residue of such a line will be different for different fields; it will depend on the constant γ; if $\varrho = \varepsilon + \varkappa\sqrt{-1}$ is its value for $\gamma = 0$ its value for an arbitrary γ will be, in consequence of (7.6).

$$
\begin{aligned}
(10.1)\qquad & \varepsilon\cos\gamma + \varkappa\sqrt{-1}\sin\gamma\sqrt{-1} + \varkappa\sqrt{-1}\cos\gamma + \varepsilon\sin\gamma\sqrt{-1} \\
& \qquad = (\varepsilon + \varkappa\sqrt{-1})(\cos\gamma + \sin\gamma\sqrt{-1}) = \varrho e^{i\gamma}.
\end{aligned}
$$

All these numbers have the same modulus $|\varrho| = \sqrt{\varepsilon^2 + \varkappa^2}$, so that we may say that only the modulus of the residue is determined by the curvature field.

If we have but one singular line (one electron) we can so choose the constant γ as to make the residue real; or, we may say, among the *possible* fields there is just one (or, more precisely, two of opposite signs) for which the magnetic charge vanishes. We can agree always to choose this field as the *existing* electromagnetic field; by this two difficulties would be solved at one stroke; the electromagnetic field would be entirely determined by the curvature field, and the fact that the magnetic charge is zero would be explained, as the result of our agreement.

But there exists more than one electron; if we have several singular lines the situation is not as simple as in the case of one singular line. If we choose our constant γ so as to make the imaginary part of the residue of one line zero we do not see immediately why the imaginary parts of the residues of other lines also should vanish; in other words, why the arguments of all residues should have values differing only by multiples of π. But we know from experimental physics that there are no magnetic charges; that means, that the existing electromagnetic field (i. e., one of the possible electromagnetic fields) has only real residues and from (10.1) it follows then, since ϱ is real, that for the possible field which corresponds to the value γ of the constant the argument is either γ or $\pi + \gamma$. It is important to notice that this experimental fact that *the differences between the arguments of the*

residues of different singular lines in every possible field is a multiple of π is a property of the space-time, because the totality of possible fields is given by the curvature field of space-time. There may be a question whether this fact can be accounted for on the general theory of relativity as it is now (i. e., whether it is a consequence of the conditions which must be satisfied by the curvature field of space-time, and which we found by eliminating the electromagnetic tensor from the energy relation and the Maxwell equations, viz. (4.71), (4.8), (8.4) and (8.5)) or whether it must be taken as an additional assumption; however this may be (see the next section) we have to consider the underlined statement as expressing an established property of space-time; but then we can determine the electromagnetic field which corresponds to a given space-time by the condition that its residues must be real. The result is the same as in the case of only one singular line. We have thus proved our contention that, under the assumptions which we have made, the electromagnetic field is entirely determined by the curvature field of space-time. These assumptions are the following:

1. In no region do the invariants of the electromagnetic field *strictly* vanish.
2. The underlined statement above.

11. Non-linearity of the field and possible consequences

Before we treat in the next section a simple example illustrating the foregoing general discussion, we cannot help indicating some speculative reasonings which bear on the assumptions just mentioned.

Considering the *analogy* with the theory of functions it may be hoped that the first of these assumptions will be deduced from the equations of the field (besides, this assumption may not be necessary because the treatment of the second canonical form of the electromagnetic tensor (3.31) may lead to the same results).

As to the second assumption there may be hope of throwing some light on it by the consideration of an essential *difference* which exists between the theory of the electromagnetic field and the theory of analytic functions. This difference is given by the fact that the conditions which define the electromagnetic field of the general relativity theory are *not linear* (see "First Note", p. 125); therefore we connot, if two different fields are given, obtain, in general, a new field by adding, say, the components in the corresponding points. In the case of analytic functions, and also in the case of electromagnetic fields of special relativity theory, we may obtain a field with two singularities by adding two fields, each of which has one singularity; there can be, in this case, no necessary connection, no interdependence between two singularities of a field, because we can choose the constants characterizing the singularities in the two fields, which are being

added, quite arbitrarily, independently each of the other. Not so in the case of the electromagnetic field of the general relativity theory; we cannot add here two fields with given singularities and be sure that the result is again a field which satisfies our conditions; the fields which are being added must satisfy some additional condition if their sum is to be such a field, and there seems to be nothing impossible in the assumption that this additional condition may bear on the constants which characterize the singularities, for instance, that it may lead to the result that the arguments of the residues can differ only by a multiple of π—and, moreover, that the moduli of the residues are equal; this would account for the equality of charges of different electrons. This additional condition may even affect the paths, i. e., the shape of singular lines.

To make this speculation more concrete we may consider two spaces given by their g's; the equations (4.7), (4.81), (8.4), (8.5) which must be satisfied by the curvature tensor field will give us equations of the second and fourth order in the g's and these equations are *not linear* in the g's. Suppose now each system of the g's defines a space with one singularity but involves arbitrary constants; if we add the corresponding g's and determine a new space by the sums, we will have some additional condition which must be satisfied by the two systems of the g's and this condition may result in relations between the constants of the two systems of the g's which are being added. Of course all this must be worked out in full detail and cannot be considered at the present time as being more than a vague suggestion.

Meanwhile we are able to treat by the preceding method only the simplest case of one singular line; we will see in the next section that we come thus to a solution which has already been obtained several times by different methods.

12. The centro-symmetric solution

We shall try to find a centrosymmetric field which satisfies our equations. In this case the expression for the line element can be taken in the form

$$-ds^2 = \xi(r)\,dr^2 + r^2\,d\vartheta^2 + r^2 \sin\vartheta \cdot d\psi^2 - \eta(r)\cdot dt^2,$$

and the mixed components of the contracted Riemann tensor are, according to the calculations of F. Kottler (Annalen der Physik, vol. 56 (1918), p. 433),

$$\begin{aligned} F_1^1 &= \frac{1}{r^2}\left(1-\frac{1}{\xi}\right) - \frac{\eta'}{\xi\eta}\cdot\frac{1}{r}, \qquad F_4^4 = \frac{1}{r^2}\left(1-\frac{1}{\xi}\right) + \frac{\xi'}{\xi^2}\cdot\frac{1}{r}, \\ F_2^2 = F_3^3 &= \frac{\xi'}{2\xi^2}\cdot\frac{1}{r} - \frac{\eta'}{2\xi\eta}\cdot\frac{1}{r} - \frac{\eta''}{2\xi\eta} + \frac{\eta'^2}{4\xi\eta^2} + \frac{\xi'\eta'}{4\xi^2\eta}, \end{aligned} \tag{12.1}$$

all the other components being zero. We see that the 2,3 plane is a plane of invariable directions with the characteristic number $F_2^2 = F_3^3$; this plane being a space-plane we must have $F_2^2 = F_3^3 = -\omega^2$ and, if our geometric conditions (4.43) are to be satisfied, the perpendicular plane must also be a plane of invariable directions and have $+\omega^2$ for its characteristic number; both F_1^1 and F_4^4 must therefore be equal to ω^2 so that we have

$$(12.2) \qquad F_1^1 = F_4^4, \qquad F_1^1 + F_2^2 = 0.$$

The same result could have been obtained algebraically: equation (4.71) shows that the square of each of the numbers F_i^i is equal to ω^4 and (4.8) that the sum of these numbers is zero; since we know that $F_2^2 = F_3^3$, we conclude that F_1^1 and F_4^4 must be equal to each other and have the sign opposite to that of $F_2^2 = F_3^3$. The equation $F_1^1 = F_4^4$ gives

$$(12.3) \qquad \frac{\xi'}{\xi} + \frac{\eta'}{\eta} = 0,$$

whence

$$(12.4) \qquad \xi\eta = 1,$$

where we gave the value 1 to the constant of integration by choosing appropriately the unit of time. If we use (12.3) and (12.4), the last two terms of the expression for F_2^2 (see (12.1)) destroy each other and the first two can be written as $-\eta'/r$; the equation $F_1^1 + F_2^2 = 0$ takes the form

$$\frac{1}{r^2}(1-\eta) - \eta' \cdot \frac{1}{r} - \eta' \cdot \frac{1}{r} - \frac{1}{2}\eta'' = 0 \qquad \text{or} \qquad \left(\frac{\eta r^2}{2}\right)'' = 1,$$

whence

$$(12.5) \qquad \eta = 1 + \frac{b}{r} + \frac{a}{r^2}, \qquad \xi = \frac{1}{1 + \frac{b}{r} + \frac{a}{r^2}}.$$

We obtain thus the known solution representing the line-element corresponding to a point charge, found for the first time by Weyl (Annalen der Physik, vol. 54 (1917), p. 117) and then by Nordstrom, Jeffery and others.

Substituting the expressions for ξ and η in the first of (12.1) we find

$$\omega^2 = F_1^1 = \frac{a}{r^4}.$$

The next task is to find the vector q. Somewhat lengthy but elementary calculations lead to the result, which is practically evident geometrically, that in our case $q = 0$; φ is therefore an arbitrary constant and the electromagnetic tensor is

$$f(x) = \frac{\sqrt{-2a}}{r^2}\sin\varphi\{i(jx) - j(ix)\} + \frac{\sqrt{2a}}{r^2}\cos\varphi\{k(lx) - l(kx)\},$$

where k, l are two mutually perpendicular unit vectors which are perpendicular to the line joining the point considered with the electron, j is a unit vector in the direction of that line and i the unit time-vector; it is clear that the residue will be real if we choose $\varphi = 0$; in this case the only components which are different from zero are

$$f_2^3 = -f_3^2 = \frac{\sqrt{2a}}{r^2},$$

the indices 2 and 3 corresponding, as above, to the coördinates ψ and ϑ, and an integration over a sphere shows that a is proportional to the square of the charge, but this, of course, is very well known. Incidentally, a is thus proportional to the square of the modulus of the residue.

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